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Preface

This volume contains 14 selected papers induced by the Conference on Differential and Difference Equations and their Applications (CDDEA 2008) held in the nice historical village Strečno, Slovak Republic, 23rd–27th June 2008. This international Conference was the 20th continuation of the previous fourteen Summer Schools on Differential Equations, the first of which was organized in 1964, and five International Conferences. The founders of the tradition were university professors Pavol Marušiak and Ladislav Berger.

The conference was a worthy continuation of the tradition and was organized by the Faculty of Science, University of Žilina. In the work of the conference 80 participants from 24 countries and 4 continents participated.


The programme contained 17 invited lectures, 42 contributed talks and 14 posters, and covered a broad part of mathematics connected with differential and difference equations and their applications and was divided into five sections (Ordinary differential equations, Functional differential equations, Difference equations, Partial differential equations and Numerical methods in differential and difference equations).

The conference was a successful and fruitful meeting stimulating scientific contacts and collaborations during nice time in Strečno.

Josef Diblík and Miroslava Růžičková
Guest editors
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WEAKLY PERTURBED NONLINEAR BOUNDARY-VALUE PROBLEM IN CRITICAL CASE

I. BOYCHUK, O. STARKOVA AND S. TCHUJKO

ABSTRACT. We construct necessary and sufficient conditions for the existence of solution of weakly nonlinear boundary value problem for a system of ordinary differential equations in critical case.

1. Statement of the Problem

We construct necessary and sufficient conditions for the existence of solution $z(t, \varepsilon)$, $z(t, \cdot) \in C^{1}[a, b]$, $z(t, \cdot) \in C[0, \varepsilon_0]$ of weakly nonlinear boundary-value problem for a system of ordinary differential equations [1, 2]

$$\frac{dz}{dt} = A(t)z + f(t) + \varepsilon Z(z, t, \varepsilon), \quad \ell z(\cdot, \varepsilon) = \alpha + \varepsilon J(z(\cdot, \varepsilon), \varepsilon). \quad (1)$$

We seek a solution of the problem (1) in a small neighborhood of the generating problem

$$\frac{dz_0}{dt} = A(t)z_0 + f(t), \quad \ell z_0(\cdot) = \alpha, \quad \alpha \in \mathbb{R}^m. \quad (2)$$

Here $A(t)$ is an $(n \times n)$ matrix, $f(t)$ is an $n$ dimensional column vector whose elements are real functions continuous on the segment $[a, b]$ and $\ell z(\cdot)$ is a linear bounded vector functional of the form $\ell z(\cdot) : C[a, b] \to \mathbb{R}^m$.

The nonlinearities $Z(z, t, \varepsilon)$ and $J(z(\cdot, \varepsilon), \varepsilon)$ are twice continuously differentiable with respect to the unknown $z$ in a small neighborhood of the generating solution and are continuous in the small parameter $\varepsilon$ in a small positive neighborhood of zero. In addition, we assume that the vector function $Z(z, t, \varepsilon)$ is continuous in the independent variable $t$ on the segment $[a, b]$. We investigate the critical case $(P_{Q\ell \neq 0})$ and the condition

$$P_{Q_2} \left\{ \alpha - \ell K[f(s)](\cdot) \right\} = 0; \quad (3)$$

is considered to be carried out, where the generating boundary-value problem (2) has an $r$-parameter family of solutions

$$z_0(t, c_r) = X_r(t)c_r + G[f(s); \alpha](t), \quad c_r \in \mathbb{R}^r, r = n - n_1.$$
Here $X(t)$ is the normal ($X(0) = I_n$) fundamental matrix of the homogeneous part of the system (2), $Q = tX(\cdot)$ is the $(m \times n)$ matrix, rank $Q = n_1$, $X_\varepsilon(t) = X(t)P_{Q^\varepsilon}$, $P_{Q^\varepsilon}$ is a $(n \times r)$ matrix composed of $r$ linearly independent columns of the $(n \times n)$ matrix (orthoprojector) $P_{Q} : \mathbb{R}^n \to N(Q)$, $P_{Q^\varepsilon}$ is an $(d \times m)$ matrix composed of $d$ linearly independent rows of the $(m \times m)$ matrix (orthoprojector) $P_{Q^\varepsilon} : \mathbb{R}^m \to N(Q^\varepsilon)$.

$$G[f(s); \alpha](t) = K[f(s)](t) - X(t)Q^+tK[f(s)](\cdot)$$
is the generalized Green operator of the boundary-value problem (2) and

$$K[f(s)](t) = X(t)\int_0^t X^{-1}(s)f(s)ds$$
is the Green operator of the Cauchy problem (2). $Q^+$ is the pseudoinverse Moore-Penrose matrix [1, 2, 3]. The lemmas below gives a condition for the existence of a solution of the problem (1) in the critical case [1, 2].

**Lemma 1.1.** Suppose that the boundary-value problem (1) corresponds to the critical case ($P_{Q^\varepsilon} \neq 0$) and condition (3) of the solvability of the problem (2) is satisfied. Also assume that problem (1) has a solution $z(t, \varepsilon) = z_0(t, c_\varepsilon) + x(t, \varepsilon)$ that turns into the generating solution $z_0(t, c_\varepsilon^*)$ for $\varepsilon = 0$. Then the vector $c_\varepsilon^* \in \mathbb{R}^r$ satisfies the equation

$$F_0(c_\varepsilon) = P_{Q^\varepsilon}\{J(z_0(\cdot, c_\varepsilon), 0) - tK[Z(z_0(s, c_\varepsilon), s, 0)](\cdot)\} = 0. \quad (4)$$

Assume that equation (4) has real roots. Fixing one of the solutions $c_\varepsilon^* \in \mathbb{R}^r$ of equation (4), we seek a solution of the problem (1) $z(t, \varepsilon) = z_0(t, c_\varepsilon^*) + x(t, \varepsilon)$ in the neighborhood of the generating solution $z_0(t, c_\varepsilon^*) = X_\varepsilon(t)c_\varepsilon^* + G[f(s); \alpha](t)$.

Perturbation $x(t, \varepsilon)$ defines a boundary-value problem

$$\frac{dx(t, \varepsilon)}{dt} = A(t)x + \varepsilon Z(z_0(t, c_\varepsilon^*) + x(t, \varepsilon), t, \varepsilon),$$

$$\ell x(\cdot, \varepsilon) = \varepsilon J(z_0(\cdot, c_\varepsilon^*) + x(\cdot, \varepsilon), \varepsilon). \quad (5)$$

In the neighborhood of the points $x = 0, \varepsilon = 0$ the following equality is true:

$$Z(z_0(t, c_\varepsilon^*) + x(t, \varepsilon), t, \varepsilon) = Z(z_0(t, c_\varepsilon^*), t, 0) + A_1(t)x(t, \varepsilon) + \varepsilon A_2(t) + R_1(z_0(t, c_\varepsilon^*) + x(t, \varepsilon), t, \varepsilon),$$

where

$$A_1(t) = \frac{\partial Z(z, t, \varepsilon)}{\partial z} \bigg|_{z = z_0(t, c_\varepsilon^*)}, \quad A_2(t) = \frac{\partial Z(z, t, \varepsilon)}{\partial \varepsilon} \bigg|_{z = z_0(t, c_\varepsilon^*)}. $$

Similarly we separate the linear part $\ell_1 x(\cdot, \varepsilon)$ with respect to $x$ and the linear part $\varepsilon \ell_2(z_0(\cdot, c_\varepsilon^*))$ with respect to $\varepsilon$ of the functional $J(z_0(\cdot, c_\varepsilon^*) + x(\cdot, \varepsilon), \varepsilon)$ and
also a term \( J(z_0(\cdot, c^*_r), 0) \) of zero order with respect to \( \varepsilon \) in the neighborhood of the points \( x = 0 \) and \( \varepsilon = 0 \)

\[
J(z_0(\cdot, c^*_r) + x(\cdot, \varepsilon), \varepsilon) = J(z_0(\cdot, c^*_r), 0) + \varepsilon \ell_1 x(\cdot, \varepsilon) + \varepsilon \ell_2(z_0(\cdot, c^*_r)) + J_1(z_0(\cdot, c^*_r) + x(\cdot, \varepsilon), \varepsilon).
\]

2. Sufficient conditions.

Defining \((d \times r)\) matrix

\[
B_0 = P_{Q_2} \{ \ell_1 X_r() - \ell K [A_1(s)X_r(s)] () \}
\]

we come to the operator system, which is equivalent to the problem of the finding of the problems’ solution (5)

\[
x(t, \varepsilon) = X_r(t)c_r + x^{(1)}(t, \varepsilon),
\]

\[
B_0 c_r = -P_{Q_2} \{ \ell_1 x^{(1)}(\cdot, \varepsilon) + \varepsilon \ell_2(z_0(\cdot, c^*_r), 0) + J_1(z_0(\cdot, c^*_r) + x(\cdot, \varepsilon), \varepsilon)
- \ell K [A_1(s)x^{(1)}(s, \varepsilon) + \varepsilon A_2(s) + R_1(z_0(s, c^*_r) + x(s, \varepsilon), s, \varepsilon)] () \},
\]

\[
x^{(1)}(t, \varepsilon) = \varepsilon G[Z(z_0(s, c^*_r) + x(s, \varepsilon), s, \varepsilon); J(z_0(\cdot, c^*_r) + x(\cdot, \varepsilon), \varepsilon)](t). \quad (6)
\]

For the construction of a solution of the boundary-value problem (5) in the critical case traditionally [1, 4] the condition \( P_{B_0} = 0 \) is used, which guarantees decidability of the second equation of the operator system (6); here \( P_{B_0} \) is an \((d \times d)\) orthoprojector matrix, \( R^d \to N(B_0^*) \).

As it is shown in the [1] there are boundary value-problems for which the condition \( P_{B_0} = 0 \) is not satisfied. For the simplification of the computations in the case \( P_{B_0} \neq 0 \) let’s assume that

\[
\frac{\partial}{\partial \varepsilon} \left[ \frac{\partial Z(z, t, \varepsilon)}{\partial z} \right] z = z_0(t, c^*_r) = 0, \quad \frac{\partial}{\partial \varepsilon} \left[ \frac{\partial J(z, t, \varepsilon)}{\partial z} \right] z = z_0(t, c^*_r) = 0.
\]

In this case in a small convex neighborhood of the points \( x = 0 \) and \( \varepsilon = 0 \) have a place the expansions [5]

\[
Z(z_0(t, c^*_r) + x(t, \varepsilon), t, \varepsilon) = Z(z_0(t, c^*_r), t, 0) + A_1(t)x(t, \varepsilon) + \varepsilon A_2(t)
+ \varepsilon^2 A_3(t) + \varepsilon A_4(t)x(t, \varepsilon) + R_2(z_0(t, c^*_r) + x(t, \varepsilon), t, t, \varepsilon), \quad (7)
\]
\[ J(z_0(\cdot, c_r^*) + x(\cdot, \varepsilon), \varepsilon) = J(z_0(\cdot, c_r^*), 0) + \varepsilon \ell_1 x(\cdot, \varepsilon) + \varepsilon \ell_2(z_0(\cdot, c_r^*), 0) + \varepsilon^2 \ell_3(z_0(\cdot, c_r^*), 0) + \varepsilon \ell_4 x(\cdot, \varepsilon) + J_2(z_0(\cdot, c_r^*) + x(\cdot, \varepsilon), \varepsilon), \]

where

\[
A_3(t) = \frac{\partial^2 Z(z,t,\varepsilon)}{2 \cdot \partial \varepsilon^2} \bigg|_{z = z_0(t,c_r^*)}, \quad A_4(t) = \frac{\partial Z(z,t,\varepsilon)}{\partial \varepsilon} \bigg|_{z = z_0(t,c_r^*)}
\]

accordingly \((n \times 1)\) and \((n \times n)\) are the matrices and the linear vector functionals

\[
\ell_3(z_0(\cdot, c_r^*), 0) = \frac{\partial^2 J(z(\cdot, \varepsilon), \varepsilon)}{2 \cdot \partial \varepsilon^2} \bigg|_{z = z_0, \varepsilon = 0}, \quad \ell_4 x(\cdot, \varepsilon) = \frac{\partial^2 J(z(\cdot, \varepsilon), \varepsilon)}{\partial \varepsilon} \bigg|_{z = z_0, \varepsilon = 0}
\]

Particular solution of the problem (5) is represented in the form

\[ x^{(1)}(t, \varepsilon) = \varepsilon G[Z(z_0(s, c_r^*), s, 0); J(z_0(\cdot, c_r^*), 0)](t) + x^{(2)}(t, \varepsilon); \]

here

\[ x^{(2)}(t, \varepsilon) = \varepsilon G[A_1(s)x(s, \varepsilon) + \varepsilon^2 A_3(s) + \varepsilon A_4(s)x(s, \varepsilon) + \varepsilon A_2(s) + \varepsilon A_4(s)x(s, \varepsilon) + R_2(z_0(s, c_r^*) + x(s, \varepsilon), s, \varepsilon); \ell_1 x(\cdot, \varepsilon) + \varepsilon \ell_2(z_0(\cdot, c_r^*), 0) + \varepsilon^2 \ell_3(z_0(\cdot, c_r^*), 0) + \varepsilon \ell_4 x(\cdot, \varepsilon) + J_2(z_0(\cdot, c_r^*) + x(\cdot, \varepsilon), \varepsilon)](t). \]

Let's define

\[ B_1 = P_{B_0^*} P_{Q_0^*} \{ \ell_1 G_1(\cdot) + \ell_4 X_r(\cdot) - \ell K \left[ A_1(s)G_1(s) + A_4(s)X_r(s) \right](\cdot) \} \]

is a \((d \times r)\) matrix. On conditions that

\[ F_1(c_r^*) := P_{Q_0^*} \{ \ell_1 G[Z(z_0(s, c_r^*), s, 0); J(z_0(\cdot, c_r^*), 0)](\cdot) + \ell_2(z_0(\cdot, c_r^*), 0) - \ell K \left[ A_1(s)G[Z(z_0(\cdot, c_r^*), s, 0); J(z_0(\cdot, c_r^*), 0)](s) + A_2(s) \right](\cdot) \} = 0 \]

the solution \( c_r = c_r^{(0)} + c_r^{(1)} \in N(B_0) \) of the second equation of the operator system (6) determines the equality

\[ c_r^{(0)} = -B_0^* P_{Q_0^*} \{ \ell_1 x^{(2)}(\cdot, \varepsilon) + \varepsilon^2 \ell_3(z_0(\cdot, c_r^*), 0) + \varepsilon \ell_4 x(\cdot, \varepsilon) + J_2(z_0(\cdot, c_r^*) + x(\cdot, \varepsilon), \varepsilon) - \ell K \left[ A_1(s)x^{(2)}(s, \varepsilon) + \varepsilon^2 A_3(s) + \varepsilon A_4(s)x(s, \varepsilon) + R_2(z_0(s, c_r^*) + x(s, \varepsilon), s, \varepsilon) \right](\cdot) \} \]
and the equation

\[\varepsilon B_1 c^{(1)}_t = -P_{B_1^0} P_{Q_2^*} \ell_1 x^{(3)}(\cdot, \varepsilon) + \varepsilon \ell_4 X_r(\cdot) c^{(0)}_r + \varepsilon^2 \ell_3 (z_0(\cdot, c^*_r), 0) + \varepsilon^2 G[Z(z_0(s, c^*_r), s, 0); J(z_0(\cdot, c^*_r)), 0](\cdot) + \varepsilon G_1(\cdot) c^{(1)}_r x^{(3)}(\cdot, \varepsilon)]
\]

\[+ J_2(z_0(\cdot, c^*_r) + x(\cdot), \varepsilon), \varepsilon - \varepsilon \ell K \{ A_1(s) x^{(3)}(s, \varepsilon) + \varepsilon A_4(s) X_r(s) c^{(0)}_r \}
\]

\[+ \varepsilon^2 A_3(s) + \varepsilon A_4(s) G[Z(z_0(\tau, c^*_r), \tau, 0); J(z_0(\cdot, c^*_r), 0)](s)
\]

\[+ \varepsilon A_4(s)[\varepsilon G_1(s)x^{(1)}(\cdot) + x^{(3)}(s, \varepsilon)] + R_2(z_0(s, c^*_r) + x(s, \varepsilon), s, \varepsilon) \}
\]

decidability of which guarantees the decidability of the operator system (6).

The equation (9) has a solution according to the condition $P_{B_1^0} P_{B_1^0} = 0$; where $P_{B_1^0} : R^d \rightarrow N(B_1^*)$ is orthoprojector matrix. Thus, in the case

\[P_{B_1^0} \neq 0, \quad P_{B_1^0} P_{B_1^0} = 0,
\]

the boundary-value problem (5) has at least one solution, which is defined by the operator system

\[x(t, \varepsilon) = X_r(t)(c^{(0)}_r + c^{(1)}_r) + x^{(1)}(t, \varepsilon),
\]

\[x^{(1)}(t, \varepsilon) = \varepsilon G[Z(z_0(s, c^*_r), s, 0); J(z_0(\cdot, c^*_r), 0)](t) + x^{(2)}(t, \varepsilon);
\]

\[x^{(2)}(t, \varepsilon) = \varepsilon G_1(t)c^{(1)}_r + x^{(3)}(t, \varepsilon), G_1(t) = G[A_1(s)X_r(s); \ell_1 X_r(\cdot)](t),
\]

\[x^{(3)}(t, \varepsilon) = \varepsilon G \{ A_1(s)[X_r(s)c^{(0)}_r + x^{(1)}(s, \varepsilon)] + \varepsilon A_2(s) + \varepsilon^2 A_3(s)
\]

\[+ \varepsilon A_4(s) [X_r(s)c^{(0)}_r + x^{(1)}(s, \varepsilon)] + R_2(z_0(s, c^*_r) + x(s, \varepsilon), s, \varepsilon);
\]

\[\ell_1 [X_r(\cdot)c^{(0)}_r + x^{(1)}(\cdot, \varepsilon)] + \varepsilon \ell_4 [X_r(\cdot)c^{(0)}_r + x^{(1)}(\cdot, \varepsilon)];
\]

\[\ell_2(z_0(\cdot, c^*_r), 0) + \varepsilon^2 \ell_3 (z_0(\cdot, c^*_r), 0) + J_2(z_0(\cdot, c^*_r) + x(\cdot), \varepsilon), \varepsilon \}
\]

\[c^{(0)}_r = -B_{10}^0 P_{Q'_1^*} \{ \ell_1 x^{(2)}(\cdot, \varepsilon) + \varepsilon^2 \ell_3 (z_0(\cdot, c^*_r), 0) + \varepsilon \ell_4 [X_r(\cdot)c_r + x^{(1)}(\cdot, \varepsilon)];
\]

\[J_2(z_0(\cdot, c^*_r) + x(\cdot, \varepsilon), \varepsilon) - \varepsilon \ell K \{ A_1(s)x^{(2)}(s, \varepsilon) + \varepsilon^2 A_3(s)
\]

\[+ \varepsilon A_4(s) [X_r(s)c_r + x^{(1)}(s, \varepsilon)] + R_2(z_0(s, c^*_r) + x(s, \varepsilon), s, \varepsilon) \}
\]

\[c^{(1)}_r = -B_{10}^0 P_{B_1^0} P_{Q_2^*} \{ \ell_1 x^{(3)}(\cdot, \varepsilon) + \varepsilon \ell_4 X_r(\cdot)c^{(0)}_r + \varepsilon^2 \ell_3 (z_0(\cdot, c^*_r), 0);
\]

\[\ell_4 [\varepsilon G_1(\cdot)c^{(1)}_r + x^{(3)}(\cdot, \varepsilon)] + J_2(z_0(\cdot, c^*_r) + x(\cdot, \varepsilon), \varepsilon)
\]

\[\ell K \{ A_1(s)x^{(3)}(s, \varepsilon) + \varepsilon A_4(s)X_r(s)c^{(0)}_r + \varepsilon^2 A_3(s)
\]

\[+ \varepsilon A_4(s)G[Z(z_0(\tau, c^*_r), \tau, 0); J(z_0(\cdot, c^*_r), 0)](s)
\]

\[+ \varepsilon A_4(s)[\varepsilon G_1(s)x^{(1)}(\cdot) + x^{(3)}(s, \varepsilon)] + R_2(z_0(s, c^*_r) + x(s, \varepsilon), s, \varepsilon) \}
\]

Thus, we have proved the following theorem.
Theorem 2.1. Suppose that the boundary-value problem (1) corresponds to the critical case $P_{Q*} \neq 0$ and the solvability condition (3) is satisfied for the generating problem (2). Then, under the condition (10) and $F_1(c_0^*) = 0$ for every root $c_0^* \in \mathbb{R}$ of equation (4) the problem (5) has at least one solution, which is defined by the operator system (11), and for $\varepsilon = 0$ that turns into the zero solution $x(t,0) \equiv 0$. Moreover, the problem (1) has at least one solution $z(t,\varepsilon) : z(t,\cdot) \in C^1[a,b]$, $z(t,\cdot) \in C[0,\varepsilon_0]$, for $\varepsilon = 0$ turns into the generating solution $z_0(t,c_0^*)$ of the problem (2).

For the construction of an approximate solution of the operator system (11) in the case 10 and $F_1(c_0^*) = 0$ one can use the method of simple iterations [1, 4]. The length of the segment $[0,\varepsilon_0]$, on which the method of simple iterations is used, can be estimated both by the means of the majorizing Lyapunov equations [2, 4], and directly from the condition of operator constructing, which is defined by the system (10), analogous to [6]. In a particular case, when $P_{B_0}P_{B_1} = 0$ the solution of the problem (1) is unique. Here

$$ P_{B_0} : \mathbb{R}^r \to N(B_0), \quad P_{B_1} : \mathbb{R}^r \to N(B_1) $$

$(r \times r)$ are orthoprojectors matrixes. The presence of derivatives

$$ A_2(s) \neq 0, \quad A_3(s) \neq 0, \quad \ell_2(z_0(\cdot, c_0^*), 0) \neq 0, \quad \ell_3(z_0(\cdot, c_0^*), 0) \neq 0, $$

and also the condition $F_1(c_0^*) = 0$ distinguish the proved theorem from the appropriate theorems [2, p. 193], [3, p. 42].

3. Periodic problem for the Mathieu equation.

An example of the conditions fulfillment of the proved theorem is a problem of the periodic solutions findings of the Mathieu equation [2, 4]

$$ y'' + (k^2 + \varepsilon h(\varepsilon) + \varepsilon \cos 2t) \cdot y = 0, \quad h(\varepsilon) = -\frac{1}{2} - \frac{\varepsilon}{16} + \frac{\varepsilon^2}{1536} + \cdots, $$

which leads to the form (1) for $k^2 = 1$,

$$ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Z(z,\varepsilon) = \text{col}[0,(-h(\varepsilon) - \cos 2t)z^{(n)}], \quad J(z(\cdot,\varepsilon),\varepsilon) \equiv 0. $$

So far as $Q = 0$, we get the critical case and $P_{Q*} = I_2$, $r = 2$,

$$ B_0 = \pi \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B_0^+ = \frac{1}{\pi} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad P_{B_0^+} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_{B_0} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. $$

The equation (4) determines a generating solution $y_0 = \cos t$, which meets the derivatives
let us check the fulfillment of conditions $F_1(\varepsilon) = 0$, $P_{B_2^*}P_{B_1^*} = 0$ and $P_{B_1}P_{B_2} = 0$, which guarantee a simple solvability of the given periodic problem for the Mathieu equation.

We obtained the solution of the Mathieu equation

\[
\begin{align*}
A_1(t) &= \begin{bmatrix}
\frac{1}{2} & 0 \\
-\cos 2t & 0
\end{bmatrix}, & A_2(t) &= \begin{bmatrix}
0 & 0 \\
\frac{1}{32} \cos t & 0
\end{bmatrix}, & A_4(t) &= \begin{bmatrix}
0 & 0 \\
-\frac{1}{32} & 0
\end{bmatrix}.
\end{align*}
\]

Matrixes

\[
B_1 = \frac{32}{\pi^2} \begin{bmatrix}
0 & 0 \\
-1 & 0
\end{bmatrix}, \quad B_1^* = \frac{32}{\pi^2} \begin{bmatrix}
0 & -1 \\
0 & 0
\end{bmatrix}, \quad P_{B_1} = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}, \quad P_{B_1^*} = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
y(t, \varepsilon) = \cos t + \frac{\varepsilon}{16} \cos 3t + \frac{\varepsilon^2}{768} (-3 \cos 3t + \cos 5t) \\
+ \frac{\varepsilon^3}{11764480} (220 \cos 3t + 30 \cos 5t - 15 \cos 7t + \cos 9t) \\
+ \frac{\varepsilon^4}{2383155200} (-7 350 \cos 3t + 1575 \cos 5t + 90 \cos 7t - 24 \cos 9t + \cos 11t) \\
+ \frac{\varepsilon^5}{951268147200} (86 625 \cos 3t - 75 495 \cos 5t + 6 426 \cos 7t \\
- 210 \cos 9t - 35 \cos 11t + \cos 13t)
\]

\[
+ \frac{\varepsilon^6}{426168129945600} (7 808 640 \cos 3t + 1 215 396 \cos 5t - 417 088 \cos 7t \\
+ 19600 \cos 9t + 420 \cos 11t - 48 \cos 13t + \cos 15t)
\]

\[
+ \frac{\varepsilon^7}{627676214435475936 \times 888 \times 06016743588}
(-1 968 211 441 083 579 268 857 856 \cos 3t \\
+ 350 747 430 860 537 215 320 064 \cos 5t + 22 420 507 574 425 725 435 904 \cos 7t \\
- 4 227 042 353 854 022 156 288 \cos 9t + 127 032 196 174 184 464 384 \cos 11t \\
+ 1 933 098 637 433 241 856 \cos 13t - 161 091 553 119 436 832 \cos 15t \\
+ 2 557 088 779 673 600 \cos 17t)
\]

\[
+ \varepsilon^8 \left( \frac{23 124 636 551 157 987 147 776 \cos 3t}{370 941 947 432 867 407 345 576 843 780 151} \\
+ \frac{3 526 980 679 450 162 991 104 \cos 5t}{512 825 842 298 602 222 036 730 531 340 453} \\
+ \frac{15 315 969 020 122 769 408 \cos 7t}{320 812 752 629 985 920 \cos 13t} \\
+ \frac{3 655 985 784 854 540 \cos 9t}{512 825 842 298 602 222 036 730 531 340 453} \\
+ \frac{3 714 017 305 249 057 \cos 11t}{8 265 213 476 777 635 352 188 688 560 447 248} \\
+ \frac{5 942 437 688 298 491 \cos 13t}{1 050 267 325 027 537 350 731 224 128 185 247 444}
\right)
\]
the appropriate functions

\[
\begin{align*}
\varepsilon_1^0(\cos 3t) & = 619 \cos 3t - 117 \cos 9t + \epsilon 10(619 \cos 3t - 117 \cos 9t) + \epsilon 11(68 \cos 3t - 88 \cos 9t) + \epsilon 12(17 \cos 3t - 88 \cos 9t) \\
\varepsilon_1^1(\cos 5t) & = 665 \cos 5t - 117 \cos 11t - 107 \cos 15t + \epsilon 10(326 \cos 5t - 117 \cos 11t - 107 \cos 15t) + \epsilon 11(431 \cos 5t - 117 \cos 11t - 107 \cos 15t) + \epsilon 12(431 \cos 5t - 117 \cos 11t - 107 \cos 15t) \\
\varepsilon_1^2(\cos 7t) & = 93 \cos 7t - 117 \cos 13t - 107 \cos 15t + \epsilon 10(326 \cos 7t - 117 \cos 13t - 107 \cos 15t) + \epsilon 11(431 \cos 7t - 117 \cos 13t - 107 \cos 15t) + \epsilon 12(431 \cos 7t - 117 \cos 13t - 107 \cos 15t)
\end{align*}
\]

We obtained also the solutions and the eigenfunctions of Mathieu equation, appropriate \( k = 2, 3 \).

**References**


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THE ON-LINE MONITORING OVER ATMOSPHERIC POLLUTION SOURCE ON THE BASE OF FUNCTION SPECIFICATION METHOD

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Abstract. The work presents the approach allows to sequential estimate of action intensity of atmospheric pollution source on the base of concentration measurements of pollution impurity in several stationary control points. The inverse problem was solved by means of the step-by-step regularization and the function specification method. Stable numerical approximation of unknown intensity are received. The solution is presented in the form of the digital filter.

Introduction

The most universal models for producing quantitative and qualitative fields of pollution distribution in the atmosphere are semi-empirical models [1].

Absence of initial data about action intensity of emission sources, distortion of boundary and initial conditions, the inadequate assessment of meteorological characteristics of the atmosphere leads to essential divergences between calculated and experimental data. Therefore joint solution of direct and inverse problems of impurity distribution in the atmosphere on the basis of data of impurity concentration measurements in stationary or mobile control points is represented expediently. This approach allows to expect an essential increase of accuracy of modelling calculations using mathematical models of acceptable complexity.

The two-dimensional linear turbulent diffusion equation [1] is employed for the description of processes of impurity distribution in the atmosphere (domain D)

\[ \frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(v_x q) + \frac{\partial}{\partial y}(v_y q) = \frac{\partial}{\partial x}\left(K_x \frac{\partial q}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial q}{\partial y}\right) + f(x, y) \cdot g(t), \]

at following conditions \( q \big|_{t=0} = h(x, y) \), \( q \big|_{\partial D} = 0 \), where \( q = q(x, y, t) \) — integral is over height of concentration of an impurity, \((v_x; v_y)\) — vector of wind speeds,
$\langle K_x; K_y \rangle$ — vector of coefficients of turbulent diffusion, $f(x,y)$ — function describing spatial arrangement of a pollution source (in elementary case, inside of a source the function is equal 1, and outside — 0), $g(t)$ — action intensity of source.

The direct problem consists in definition of the concentration field $q(x,y,t)$ at the known values $(v_x; v_y)$, $(K_x; K_y)$, $f(x,y)$, $g(t)$. Let’s examine the inverse problem consisting in definition of function $g(t)$ at the known values $(v_x; v_y)$, $(K_x; K_y)$, $f(x,y)$ and $q(x,y,t)$.

In practice in definition of the concentration field $q(x,y,t)$ there is a number of problems:

- concentration measurements are not taken in all domain $D$;
- concentration cannot be measured time-continuously;
- there are errors of measurements.

Knowing that we shall consider the following:

- in the points $(x_j, y_j)$, $j = 1, \ldots, J$ stationary control points are located;
- measurements are taken in identical time intervals $\Delta t$.
- error of concentration measurements is additive and has the normal distribution.

Then we can write down the formula

$$c_{ji} = q(x_j, y_j, t_i) + \delta \cdot \gamma,$$

where $c_{ji}$ — concentration measured by $j^{th}$ sensor at the time moment $t_i = i \cdot \Delta t$, $\delta$ — root-mean-square error of sensor measurements, $\gamma$ — standardized Gaussian variate ($\text{Average(\gamma)} = 0, \text{Variance(\gamma)} = 1$).

### 1. The solution of the inverse problem

The inverse problem for the source is characterized by solution instability with respect to errors of concentration measurements and demands special methods of the solution [2, 3, 4].

Such methods can be direct methods (step-by-step regularization) [2], Tikhonov regularization [3] and function specification method [4]. In the function specification method the functional form of unknown intensity is supposed in advance and includes a number of the unknown parameters estimated by means of the least-squares method. The unknown intensity can be estimated simultaneously for all interval of time $[0, T]$ (whole domain approximation) and it is sequential on each time interval $[t_{N-1}; t_N]$ (sequential approximation).

Methods of step-by-step regularization and sequential function specification were used to solve the inverse problem. The functional form at which $g(t)$ accepts each time interval $[t_{N-1}; t_N]$ constant value $g_N$ (constant piecewise approximation) is considered.
1.1. Duhamel’s theorem and sensitivity coefficients

Linearity of the problem (1) allows to use the superposition principle and numerical analogue of Duhamel’s theorem

\[ q(x_j, y_j, t_i) = Q_{init}(x_j, y_j, t_i) + \sum_{n=1}^{i} g_n \cdot (Q(x_j, y_j, t_{i-n+1}) - Q(x_j, y_j, t_{i-n})) \]  

(2)

where \( Q(x, y, t) \) — solution of the direct problem (1) at \( g(t) = 1 \) and \( q\big|_{t=0} = 0 \), \( Q_{init}(x, y, t) \) — solution of the direct problem (1) at \( g(t) = 0 \) and \( q\big|_{t=0} = h(x, y) \).

Let’s enter designations \( Q(x, y, t_i) = \phi_{ji} \), \( \phi_{j(i+1)} - \phi_{ji} = \Delta \phi_{ji} \).

The magnitude \( \phi_{ji} \) is called step sensitivity coefficient. It represents impurity concentration in a point of an arrangement \( j^{th} \) sensor at the moment of time \( t_i \), at condition \( g(t) = 1 \). The magnitude \( \phi_{ji} \) is possible to present as the response to unit step of intensity of a source, an event at the moment of time \( t_0 = 0 \). Then the magnitude \( \Delta \phi_{ji(n)} \) can be treated as impurity concentration in a point of an arrangement \( j^{th} \) sensor at the moment of time \( t_i \) provided that unit step of intensity of a source has happened at the moment of time \( t_n \).

The magnitude \( \Delta \phi_{ji(n)} \) is called pulse sensitivity coefficient and represents itself an increment of impurity concentration in a point of an arrangement \( j^{th} \) sensor at the moment of time \( t_i \), at unit impulse (duration \( \Delta t \)) of intensity of a source which was proceeding during the time interval \([t_{n-1}; t_n]\).

1.2. Sequential function specification method

We shall estimate \( g_N \), considering \( g_1, g_2, \ldots, g_{N-1} \) are known values, calculated on the previous steps. To give stability to the solution of the inverse problem we shall consider \( g(t) \) on several \( (r) \) time intervals at once. Let’s consider, that \( g_N, g_{N+1}, \ldots, g_{N+r-1} \) are connected by some functional dependence. In case of \( r = 1 \) the method step-by-step regularization turns out.

Using (2) for the moments of time \( t_N, t_{N+1}, \ldots, t_{N+r-1} \) let’s write down the matrix equation

\[ Q = Q_{init} + Q_{ig=0} + \Phi \cdot G \]

(3)

where

\[ Q = \begin{pmatrix} Q(N) \\ Q(N+1) \\ \cdots \cdots \\ Q(N+r-1) \end{pmatrix}, \quad Q(i) = \begin{pmatrix} q(x_1, y_1, t_i) \\ q(x_2, y_2, t_i) \\ \cdots \cdots \\ q(x_j, y_j, t_i) \end{pmatrix}, \quad G = \begin{pmatrix} g_N \\ g_{N+1} \\ \cdots \cdots \\ g_{N+r-1} \end{pmatrix} \]

\[ Q_{init} = \begin{pmatrix} Q_{init}(N) \\ Q_{init}(N+1) \\ \cdots \cdots \\ Q_{init}(N+r-1) \end{pmatrix}, \quad Q_{init}(i) = \begin{pmatrix} Q_{init}(x_1, y_1, t_i) \\ Q_{init}(x_2, y_2, t_i) \\ \cdots \cdots \\ Q_{init}(x_j, y_j, t_i) \end{pmatrix} \]
The equation (3) can be solved exactly only for the case \( r = 1 \) and \( J = 1 \) (Stolz solution). In this case the solution of the inverse problem is frequently unstable. In the case of using several time steps \( (r > 1) \) or several sensors \( (J > 1) \) the equation can be solved only approximately by means of the least-squares method.

Let’s minimize the sum of squares of differences between measured \( C \) and calculated \( Q \) values of concentration

\[
S = (C - Q)^T \cdot (C - Q) \rightarrow \min, \tag{4}
\]

where

\[
C = \begin{pmatrix} C(N) \\ C(N+1) \\ \cdots \\ C(N+r-1) \end{pmatrix}, \quad C(i) = \begin{pmatrix} c_{1i} \\ c_{2i} \\ \cdots \\ c_{ji} \end{pmatrix}.
\]

It is possible to approach the solution of the equation (3) in two ways:

- to solve equations set with \( r \) unknown values \( g_N, g_{N+1}, \ldots, g_{N+r-1} \);
- to reduce number of unknown values considering, that \( g_{N+1}, g_{N+2}, \ldots, g_{N+r-1} \) is expressed by mean of some functional dependence from \( g_N \) and from the previous values \( g_{N-1}, g_{N-2}, \ldots, g_{N-p} \).

In the first case values \( g_N, g_{N+1}, \ldots, g_{N+r-1} \) can turn out unrelated values themselves, though in practice values of intensity \( g(t) \) cannot be vary arbitrarily.

In the second case the chosen functional dependence provides improvement of smoothness of the solution.

Let this functional dependence looks like

\[
G = A \cdot g_N + B \cdot G_0, \tag{5}
\]

where

\[
G_0 = \begin{pmatrix} g_{N-1} \\ g_{N-2} \\ \cdots \\ g_{N-p} \end{pmatrix}, \quad A, B \text{ are the } r \times 1 \text{ and } r \times p \text{ matrixes.}
\]

Then the sequential estimation algorithm will look like:
1. for the chosen functional dependence $g_{N+1}$, $g_{N+2}$, ..., $g_{N+r−1}$ from $g_N$ and $g_{N−1}$, $g_{N−2}$, ..., $g_{N−p}$ we shall estimate the unique unknown value $g_N$;
2. we shall pass to a following step, temporarily assuming dependence $g_{(N+1)+1}$, $g_{(N+1)+2}$, ..., $g_{(N+1)+r−1}$ from $g_{(N+1)}$ and $g_{(N+1)−1}$, $g_{(N+1)−2}$, ..., $g_{(N+1)−p}$.

We examine the elementary case of functional dependence — the assumption of a constancy $g_N$ during $r$ the sequential intervals of time (constant intensity approximation)

$$g_N = g_{N+1} = \cdots = g_{N+r−1},$$

in this case

$$A = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad p = 1, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Also we examine the case of linear dependence between $g_N$, $g_{N+1}$, ..., $g_{N+r−1}$ (linear intensity approximation)

$$g_{N+i−1} = g_N + (i−1) \cdot (g_N − g_{N−1}) = i \cdot g_N + (1−i) \cdot g_{N−1},$$

in this case

$$A = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ r \end{pmatrix}, \quad p = 1, \quad B = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ r−1 \end{pmatrix}.$$

Using (5) in (3) we have

$$Q = Q_{\text{init}} + Q_{g=0} + \Phi \cdot B \cdot G_0 + \Phi \cdot A \cdot g_N.$$

Employing the last expression in (4) and having calculating the matrix derivative, we can calculate the intensity estimation $g_N$

$$\hat{g}_N = \left( X^T \cdot X \right)^{-1} \cdot X^T \cdot \left( C - Q_{\text{init}} - Q_{g=0} - \Phi \cdot B \cdot G_0 \right), \quad (6)$$

where $X = \Phi \cdot A$.

1.3. The solution in the digital filter form

The solution (6) is a linear function of the measured concentration $c_{ji}$, $i = 1, 2, \ldots, N+r−1$ and it is possible to present in the form of the digital filter [5]

$$\hat{g}_N = \sum_{i=1}^{N+r−1} \sum_{j=1}^J f_{ji(N−i)} \cdot (c_{ji} - Q_{\text{init}}(x_j, y_j, t_i)),$$

where $f_{ji(i−r)}$ — coefficients of the filter, $f_{ji(i−r)} = g_{ji}$, $i = 1, 2, \ldots, N+r−1$, $g_{ji}$ — solution of the inverse problem (6) at conditions $c_{jr} = 1$; $c_{ji} = 0$, $i \neq r$; $Q_{\text{init}}(x, y, t) = 0$.

The solution in the form of the digital filter is more effective than other forms in the computing relation because coefficients $f_{jk}$ are calculated once.
2. Computing experiments

For the solution of the inverse problem it is necessary to know sensitivity coefficients which are calculated from the solution of the direct problem at \( g(t) = 1 \). For the solution of the direct problem the program is written in programming language Fortran. In the program was used implicit difference scheme. For modelling concentration measurements the direct problem is solved at known function of action intensity of a source \( g(t) \). The solution of the inverse problem and visualization of calculations are realized in mathematical package MatLab.

Parameters of algorithm of the inverse problem solution are:
- \( \Delta t \) — step of concentration measurements;
- \( r \) — quantity of sequential time steps;
- matrices \( A, B \) define the form of functional approximation;
- \( J \) — quantity of sensors;
- \((x_j, y_j)\) — position of sensors (influences of sensitivity coefficients);
- \( \delta \) — root-mean-square error of measurements of the sensor.

2.1. Graphs of sensitivity coefficients

The pulse sensitivity function \( \Delta Q(x, y, t) = Q(x, y, t) - Q(x, y, t - \Delta t) \) represents the response on unit impulse (during time \( \Delta t \)) change of action intensity of a source (see figure 1). The pulse sensitivity coefficients (PSC) represent discrete samples of pulse sensitivity function in the equidistant moments of time (see figure 2) \( \Delta \phi_{ji} = Q(x_j, y_j, t_{i+1}) - Q(x_j, y_j, t_i) = \Delta Q(x_j, y_j, (i+1)\Delta t) \).

Graphs of sensitivity functions have one maximum. In figure 1 the phenomena of lag and damping are clearly visible. The sensor number 1 is most sensitive: it is faster (lag less, than at others sensors) and stronger (damping less, than at other sensors) reacts to intensity change of a source.

If we arrange the sensor in an operative range of a source then there is not lag and damping practically is not observed (in this case we have the pseudo-inverse problem). Increasing distance from a source the effects of lag and damping are getting more strongly: the maximum of sensitivity function moves to the right, the maximal value of function decreases, the graph becomes more flat.
2.2. The step-by-step regularization method

In case of step-by-step regularization \((r = 1)\) the matrixes \(A, B\) degenerate to the \(1 \times 1\) matrixes: \(A = (1), B = (0)\).

The solution (6) looks like

\[
\hat{g}_N = \left(\phi(0)^T \cdot \phi(0)\right)^{-1} \cdot \phi(0)^T \cdot \left(C(N) - Q_{\text{init}}(N) - Q|_{g=0}(0)\right),
\]

or in the unwrapped kind (considering, that \(\Delta \phi_{j1} = \phi_{j1}\))

\[
\hat{g}_N = \sum_{j=1}^{J} \phi_{j1} \cdot \left( c_{jN} - Q_{\text{init}}(x_j, y_j, t_N) - \sum_{n=1}^{N-1} g_n \cdot \Delta \phi_{j(N-n)}\right).
\]

If we use one sensor we shall receive the Stolz solution

\[
\hat{g}_N = \frac{c_{1N} - Q_{\text{init}}(x_1, y_1, t_N) - \sum_{n=1}^{N-1} g_n \cdot \Delta \phi_{1(N-n)}}{\phi_{11}}.
\]

It is necessary for stability of the Stolz solution to have as big as possible the pulse sensitivity coefficient \(\Delta \phi_{10} = \phi_{11}\). Let’s allow jump of intensity happened at the moment of time \(t = 0\). It is better to make measurements at the moment of time when the response to this jump is maximal. It is necessary to choose the step of concentration measurements \(\Delta t\) so, that \(\Delta \phi_{10} = \Delta \phi_{11}\) (see figure 2).

In figure 3 there are the examples of an estimation of intensity at an identical step \(\Delta t\) for three cases:

1. function \(g(t)\) is smooth; there aren’t errors of measurements;
2. function \(g(t)\) has jump; there aren’t errors of measurements;
3. function \(g(t)\) has jump; there are errors of measurements \((\delta = 0.01 \cdot q_{\text{max}})\).

The magnitude \(q_{\text{max}}\) is the maximal concentration measured by sensor. Here and there in figures it is shown the true function of intensity \(g(t)\) by dashed line and the estimation of intensity by continuous line.

![Figure 3.](image)

Even in case of the pseudo-inverse problem (the sensor is located in an operative range of a source or on border of a source) it is possible to have fluctuations of the solution at presence the errors of measurements, but the solution remains stable.
For each sensor there is the step $\Delta t_{st[1]}$ (the minimal step) at which solution of the inverse problem is stable.

### 2.3. The preliminary filtration and the post-filtration

To reduce fluctuations of the solution it is possible to execute the preliminary filtration of input data (measured concentration) and the post-filtration of output data (estimated values of intensity). Results of the preliminary filtration and the post-filtration are shown on figure 4. The filtration was made by means of nonrecursive filter [5]

$$y_i = \frac{x_{i-1} + 2x_i + x_{i+1}}{4}.$$

![Figure 4.](image)

**Figure 4.** $r = 1, \delta = 0.03 \cdot q_{max}$. The filtration was not applied (left); the preliminary and the post-filtration was applied (right).

### 2.4. The function specification method

Stability of the solution of the inverse problem can achieve at the step of measurements $\Delta t < \Delta t_{st[1]}$, using several sequential time steps ($r > 1$). The value $\Delta t_{st[r]}$ (the minimal time step at which the solution of the inverse problem is stable) decreases with increasing $r$ (see figure 2.4).

### 2.5. About accuracy of estimation of desired intensity

The root-mean-square error was used for the account of accuracy of intensity estimation $g(t)$

$$\sigma_G = \sqrt{\frac{1}{N_{max}} \sum_{n=1}^{N_{max}} (g \left( n - \frac{1}{2} \right) \cdot \Delta t) - \hat{g}_n)^2}.$$

It is experimentally possible to choose the time step $\Delta t_{opt}$ at which $\sigma_G$ is minimal.

It is observed that at $r = 1$ with increase $\Delta t$ dependence of value $\sigma_G$ from parameter $\delta$ weakens (see figure 6). At $\Delta t = \text{const}$ with increase $r$ influence of parameter $\delta$ on value $\sigma_G$ weakens and at certain $r = r_c$ the value $\sigma_G$ practically does not depend from $\delta = 0 \div 0.03 \cdot q_{max}$ (see figure 7).
3. Results of computing experiments

The numerous quasi-real experiments are lead using number of methodical problems. Stable numerical approximation to desired value intensity for sources of various types (point, linear, areal, distributed) are constructed, at presence of
measurement errors in sensors (parameter $\delta = 0 \div 0.03 \cdot q_{\text{max}}$). Sensors are settled down outside of an operative range of a source ($f(x, y) = 0$) and in an operative range of a source ($f(x, y) \neq 0$).

For each sensor there is the critical step $\Delta t_{st[1]}$, such, that for each step of the solution of the inverse problem $\Delta t > \Delta t_{st[1]}$ the solution is stable, i.e. the step-by-step regularization effect takes place. But its opportunities are limited, because for some sensor $\Delta t_{st[1]}$ can be big enough therefore the estimated solution becomes worse.

The desire to increase the accuracy of intensity estimation, reducing time step $\Delta t$, leads to instability of the solution of inverse problem. Using several sensors ($J > 1$) the sensor with smaller $\Delta t_{st[1]}$ has the prevailing influence. Using two sensors with identical $\Delta t_{st[1]}$ improve result in comparison with one sensor. To use functional specification method with several ($r > 1$) sequential time steps makes the solution stability more effective.

The analysis of results of numerical experiments allows to draw a conclusion, that for pair numbers $(\Delta t/\Delta t_{st[1]}, \delta)$, $\Delta t/\Delta t_{st[1]} \in [0, 1; 1]$, $\delta \in [0; 0.03 \cdot q_{\text{max}}]$ it is possible to select $r$ at which the error of intensity estimation $g(t)$ is minimal.

The data of concentration measurements from sensors is understanding sequen-

tially in the considered method, that allows to organize the on-line monitoring over emissions of pollution impurity in the atmosphere.

References

PERTURBATION PRINCIPLE IN DISCRETE HALF-LINEAR OSCILLATION THEORY

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Abstract. We investigate oscillatory properties of solutions of the second order half-linear difference equation

$$\Delta(r_k\Phi(\Delta x_k)) + c_k\Phi(x_{k+1}) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad p > 1,$$

where this equation is viewed as a perturbation of another (nonoscillatory) half-linear equation of the same form.

1. Introduction

We deal with the oscillatory properties of the second order half-linear difference equation

$$\Delta(r_k\Phi(\Delta x_k)) + c_k\Phi(x_{k+1}) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad p > 1. \quad (1)$$

Qualitative theory of (1) is summarized in the book [3, Chap. 3]. It is shown there that solutions of (1) behave in many aspects similarly as those of the linear equation

$$\Delta(r_k\Delta x_k) + c_kx_{k+1} = 0, \quad (2)$$

which is the special case $p = 2$ in (1) and whose oscillation theory is deeply developed from various points of view, we refer to the books [1, 3, 16] and the references given therein.

Another motivation for the investigation of (1) is its continuous counterpart, the half-linear differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad (3)$$

which attracted considerable attention in recent years, we refer to the books [2, 9].

In the classical approach to the oscillation theory of (3), this equation is regarded as a perturbation of the (nonoscillatory) one term equation

$$(r(t)\Phi(x'))' = 0. \quad (4)$$

In this setting, the methods used to study (3) are very similar to those used in linear oscillation theory since the solution space of (4) is linear. Recently, another

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approach has been introduced in [7], the so-called perturbation principle, where (3) is regarded as a perturbation of a (nonoscillatory) half-linear equation of the same form (whose solution space is no longer additive). In this approach, a Riccati type differential equation appears and this equation involves a certain nonlinear function. Many results obtained by perturbation principle are then based on the quadratization of this nonlinear function. We will recall basic ideas of this method in the next section.

Concerning the discrete version of the perturbation principle, the “preliminary” step has been made in [8]. There the concept of the recessive solution of the nonoscillatory half-linear difference equation
\[ \Delta(x_k \Phi(\Delta x_k)) + \tilde{c}_k \Phi(x_{k+1}) = 0 \] (5)
has been introduced and it was shown that (1), viewed as a perturbation of (5), is oscillatory provided \( \sum_{k=0}^{\infty} (c_k - \tilde{c}_k) h_{k+1} = \infty \) where \( h \) is the positive recessive solution of (5). In the proof of this statement no Riccati type difference equation appears and this proof is based on the so-called variational principle.

In the following papers [5, 6], the Riccati technique in perturbation principle has been introduced and it turned that to find a quadratic approximation of nonlinear function in this equation is considerably more complicated than in the continuous case and several problems remained open. We formulate some of these problems in last section of the paper.

The paper is organized as follows. In the next section we recall basic ideas of the perturbation principle both in the continuous and discrete case. Section 3 deals with oscillation of perturbed Euler-type differential equation, where some open problems formulated in [5] can be solved due to the special structure of the nonlinearity in modified Riccati equation. The last section is devoted to the formulation of some open problems related to our research.

2. Perturbation principle

We start with the continuous case. Together with (3) we consider the equation of the same form
\[ (r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0. \] (6)
We suppose that this equation is nonoscillatory and that \( h \) is its eventually positive solution. Put \( v = h^p(w - w_h) \), where \( w \) is a solution of the Riccati equation associated with (3)
\[ w' + c(t) + (p - 1)r^{1-q}|w|^q = 0, \quad q := \frac{p}{p - 1}, \] (7)
and \( w_h = r \Phi(h'/h) \). Then \( v \) solves the so-called modified Riccati equation
\[ v' + (c(t) - \tilde{c}(t))h^p(t) + (p - 1)r^{1-q}(t)h^{-q}(t)H(t, v) = 0, \] (8)
where
\[ H(t, v) := |v + G(t)|^q - qv\Phi^{-1}(G(t)) - |G(t)|^q \]
with \( G(t) := r(t)h(t)\Phi(h'(t)) \) and \( \Phi^{-1} \) the inverse function of \( \Phi \). There are several reasons why we call (8) the modified Riccati equation. One of them is that if
\( p = 2 \), then (8) reduces to the equation
\[
v' + (c(t) - \ddot{c}(t))h^2(t) + \frac{v^2}{r(t)h^2(t)} = 0,
\]
which is the Riccati equation corresponding to the linear equation
\[
(r(t)h^2(t)y')' + (c(t) - \ddot{c}(t))h^2(t)y = 0,
\]
which results from (3) with \( p = 2 \) upon the transformation \( x = h(t)y \) where \( h \) is a solution of (6) with \( p = 2 \). Another reason is that (8) reduces to (7) when \( h(t) \equiv 1 \).

If a solution \( h \) of (6) satisfies additionally \( h'(t) \neq 0 \), the function \( H \) can be written in the form
\[
H(t, v) = |G(t)|^q F \left( \frac{v}{G(t)} \right), \quad F(z) := |1 + z|^q - qz - 1.
\]
This formula shows that \( H(t, v) \geq 0 \) for \( v \in \mathbb{R} \) with equality if and only if \( v = 0 \), \( H_v(t, v) = 0 \) if and only if \( v = 0 \) and that \( H \) is convex in \( v \). Moreover, if \( v(t)/G(t) \to 0 \) as \( t \to \infty \) for (positive) solutions of (8), one can approximate the function \( F \) by its second degree Taylor polynomial
\[
F(z) = \frac{q(q - 1)}{2} z^2 + o(z^2) \quad \text{as} \quad z \to 0.
\]
Then, substituting this formula to (8) and neglecting the term \( o(z^2) \), one can write (8) in the “approximative” form
\[
v' + (c(t) - \ddot{c}(t))h^p(t) + \frac{q}{2R(t)} v^2 = 0, \quad R(t) := r(t)h^2(t)|h'(t)|^{p-2}.
\]

The last equation is the “classical” Riccati equation associated with the linear second order equation
\[
(R(t)x')' + \frac{q}{2} C(t)x = 0, \quad C(t) := (c(t) - \ddot{c}(t))h^p(t).
\]
This fact enables to use many “linear” results in the half-linear oscillation theory, see, e.g., [10]. Moreover, (9) and (10) can be solved explicitly in some special cases. For example, if
\[
C(t) = \frac{1}{2qR(t)(\int_{t}^{\infty} R^{-1}(s) ds)^2} \quad \text{or} \quad C(t) = 0,
\]
then \( v(t) = \frac{1}{q \int_{t}^{\infty} R^{-1}(s) ds} \) or \( v(t) = \frac{2}{q \int_{t}^{\infty} R^{-1}(s) ds} \), respectively. This fact has been used in [11] and [12] to study conditionally oscillatory half-linear equations and its so-called nonprincipal solutions. In addition to the approximation formula near \( v = 0 \) we have the following global inequalities for the last term in the left-hand side of (8) which hold for all \( v \in \mathbb{R} \):
\[
(p - 1)r^{-q}(t)h^{-q}(t)H(t, v) \leq \frac{q}{2R(t)} v^2, \quad q \geq 2,
\]
\[
(p - 1)r^{-q}(t)h^{-q}(t)H(t, v) \geq \frac{q}{2R(t)} v^2, \quad q \in (1, 2].
\]
These inequalities have been used in [4] to study integral characterization of the so-called principal solution of (3).
Now we turn our attention to half-linear difference equation (1). Oscillatory properties of (1) are defined using the concept of the generalized zero which is defined in the same way as for (2), see, e.g., [3, Chap. 3] or [9, Chap. 7]. A solution \( x \) of (1) has a generalized zero in an interval \((m, m + 1)\) if \( x_m \neq 0 \) and \( x_m x_{m+1} r_m \leq 0 \). Since we suppose that \( r_k > 0 \) (oscillation theory of (1) generally requires only \( r_k \neq 0 \)), a generalized zero of \( x \) in \((m, m + 1)\) is either a “real” zero at \( k = m + 1 \) or the sign change between \( m \) and \( m + 1 \). Equation (1) is said to be disconjugate in a discrete interval \([m, n]\) if the solution \( x \) of (1) given by the initial condition \( x_m = 0 \), \( x_{m+1} \neq 0 \) has no generalized zero in \((m, n + 1)\). Equation (1) is said to be nonoscillatory if there exists \( m \in \mathbb{N} \) such that it is disconjugate on \([m, n]\) for every \( n > m \) and is said to be oscillatory in the opposite case. This terminology typical for linear equations is correct since the discrete Sturmian theory extends almost verbatim to (1).

Similarly as in the continuous case we regard equation (1) as a perturbation of the nonoscillatory equation (5). Let \( h \) be an eventually positive solution of (5), \( \tilde{w} = r \Phi(\Delta h/h) \), and let \( w \) be a solution of the Riccati type equation associated with (1)
\[
 w_{k+1} + c_k - \frac{r_k w_k}{\Phi(\Phi^{-1}(r_k) + \Phi^{-1}(w_k))} = 0. \tag{12}
\]
Then \( v = h^p (w - \tilde{w}) \) is the solution of the difference equation
\[
 \Delta v_k + (c_k - \tilde{c}_k) h^p_{k+1} + H(k, v) = 0,
\]
where
\[
 H(k, v) := v + r_k h_{k+1} \Phi(\Delta h_k) - \frac{r_k h^p_{k+1} (v + G_k)}{\Phi(h^p_k \Phi^{-1}(r_k) + \Phi^{-1}(v + G_k))}. \tag{13}
\]
with
\[
 G_k := r_k h_k \Phi(\Delta h_k). \tag{14}
\]
Moreover, \( w_k + r_k > 0 \) (i.e., a solution of (1) which gives \( w \) via the formula \( w = r \Phi(\Delta x/x) \) has no generalized zero in \((k, k + 1]\) if and only if \( v_k + v^*_k > 0 \), where
\[
 v^*_k := r_k h_k (\Phi(h_k) + \Phi(\Delta h_k)). \tag{15}
\]
Concerning the quadratic approximation of the function \( H \) in (13), we have in disposal no global estimates like (11) in the continuous case yet, and in the next statement we need to distinguish the cases \( G_k < 0 \) and \( G_k > 0 \), see [5, 6].

**Theorem 2.1.** Let \( G \) and \( v^* \) be defined by (14) and (15), respectively, and
\[
 R_k := \frac{2}{q} r_k h_k h_{k+1} |\Delta h_k|^{p-2}. \tag{16}
\]

(i) Let \( p \in (1, 2] \). If \( G_k > 0 \) for \( k \in \mathbb{N} \), then
\[
 H(k, v) \geq \frac{v^2}{R_k + v} \quad \text{for } v \geq 0,
\]
if \( G_k < 0 \) for \( k \in \mathbb{N} \), then \( v^*_k \leq R_k \) and (17) holds for \( v \in (-v^*_k, 0] \).

(ii) Let \( p \geq 2 \). If \( G_k > 0 \), then
\[
 H(k, v) \leq \frac{v^2}{R_k + v} \quad \text{for } v \geq 0,
\]
if \( G_k < 0 \) for \( k \in \mathbb{N} \), then \( v^*_k \geq R_k \) and (18) holds for \( v \in (-R_k, 0] \).
The reason why the term $R + v$ appears in the denominator in (17), (18) is that this term appears also in the Riccati equation corresponding to linear Sturm-Liouville difference equation (substitute $p = 2$ in (1) and (12)).

As a consequence of the previous theorem we have the following statements, see [5].

**Theorem 2.2.** Let $c_k \geq \tilde{c}_k$ for large $k$.

(i) If $p \in (1, 2]$, $h$ is the recessive solution of (5), and the linear equation

$$\Delta(R_k\Delta x_k) + C_k x_{k+1} = 0,$$

where $R$ is given by (16) and

$$C_k := (c_k - \tilde{c}_k) h_k^{p+1},$$

is oscillatory, then equation (1) is also oscillatory.

(ii) If $p \geq 2$, $\sum_{k=1}^\infty R_k^{-1} = \infty$ and the linear equation (19) is nonoscillatory, then equation (1) is also nonoscillatory.

We finish this section by a theorem which does not distinguish between the cases $p \in (1, 2]$ and $p \geq 2$ which is also proved in [5]. We use this statement to prove the main result of our paper which is given in the next section.

**Theorem 2.3.** Let $c_k \geq \tilde{c}_k$ for large $k$ and $h_k > 0$ be the recessive solution of (5) such that

$$\sum_{k=1}^\infty (c_k - \tilde{c}_k) h_k^{p+1} < \infty.$$

Further suppose that the condition $\sum_{k=1}^\infty R_k^{-1} = \infty$ holds and

$$\lim_{k \to \infty} G_k = \infty, \quad G_k := r_k h_k \Phi(\Delta h_k).$$

If there exists $\varepsilon > 0$ such that equation

$$\Delta(R_k\Delta y_k) + (1 - \varepsilon)C_k y_{k+1} = 0$$

is oscillatory, then equation (1) is also oscillatory.

3. Perturbed Euler-type equation

In this section we consider the Euler type difference equation

$$\Delta(\Phi(\Delta x_k)) + \tilde{c}_k \Phi(x_{k+1}) = 0,$$

where

$$\tilde{c}_k := -\frac{\Delta \Phi(\Delta h_k)}{\Phi(h_{k+1})}, \quad h_k = k^{\frac{1}{p-1}}.$$

Our motivation comes from the continuous case where $h(t) = t^{\frac{1}{p-1}}$ is the principal solution of the Euler-type equation

$$(\Phi(x'))' + \gamma p \Phi(x) = 0, \quad \gamma_p := \left(\frac{p-1}{p}\right)^p$$

whose properties were studied in several papers, see e.g. [14, 18]. It was shown in [5] that

$$\tilde{c}_k = \frac{\gamma_p}{(k+1)^p} \left[1 + O(k^{-1})\right],$$

(27)
and that
\[ G_k = \left( \frac{p-1}{p} \right)^{p-1} \left[ 1 - \frac{p-1}{2pk} + o(k^{-1}) \right], \quad (28) \]
both as \( k \to \infty \).

In [5] we conjectured that under some technical assumption the perturbed Euler equation
\[ \Delta(\Phi(\Delta x_k)) + [\tilde{c}_k + d_k] \Phi(x_{k+1}) = 0 \quad (29) \]
is oscillatory provided
\[ \liminf_{k \to \infty} \log k \left( \sum_{j=k}^{\infty} d_j (j+1)^{p-1} \right) > \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1}. \]

In this section we prove that this conjecture is true for \( p \geq 2 \). It has been shown in the recent paper [6] that if \( p \geq 2 \), then the sequence \( h_k = k^{1/p} \) is the recessive solution of (24). This fact enables to apply Theorem 2.3 (with \( h_k = k^{1/p} \) and (24) instead of (5)) to equation (29).

Note that Theorem 2.3 cannot be applied directly to (29) because of condition (22), which is not satisfied in this case. However, going through the proof of Theorem 2.3, one can see that condition (22) can be replaced by the following two conditions
\[ \sum_{k=1}^{\infty} H(k,v) = \infty \quad \text{for every} \quad v > 0 \quad (30) \]
and
\[ \liminf_{k \to \infty} G_k > 0, \quad G_k = r_k h_k \Phi(\Delta h_k), \quad (31) \]
which are less restrictive, see [5].

We will need the following Hille-Nehari oscillation criterion for the linear equation (2).

**Lemma 3.1** ([15]). Suppose that \( c_k \geq 0 \), \( r_k > 0 \), \( \sum_{k=1}^{\infty} r_k^{-1} = \infty \), \( \sum_{k=1}^{\infty} c_k < \infty \). If
\[ \liminf_{k \to \infty} \left( \sum_{j=k}^{k-1} r_j^{-1} \right) \left( \sum_{j=k}^{\infty} c_j \right) > \frac{1}{4}, \]
then (2) is oscillatory.

**Theorem 3.2.** Let \( p \geq 2 \). Consider equation (29) with \( \tilde{c}_k \) given in (25) and suppose that \( d_k \geq 0 \) for large \( k \) and
\[ \sum_{k=1}^{\infty} d_k (k+1)^{p-1} < \infty. \quad (32) \]
Then equation (29) is oscillatory provided
\[ \liminf_{k \to \infty} \log k \left( \sum_{j=k}^{\infty} d_j (j+1)^{p-1} \right) > \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1}. \quad (33) \]
Proof. We apply Theorem 2.3 with \( c_k = \tilde{c}_k + d_k \). We have \( c_k - \tilde{c}_k \geq 0 \) for large \( k \) and, since \( p \geq 2 \), the sequence \( h_k = k^{\frac{p-1}{p}} \) is the recessive solution of (24), see [6]. Condition (32) implies

\[
\sum_{k=1}^{\infty} (c_k - \tilde{c}_k) h_k^{p+1} = \sum_{k=1}^{\infty} d_k (k+1)^{p-1} < \infty,
\]

hence (21) is satisfied. By a direct computation (see [5]) we obtain

\[
h_k h_{k+1}\Delta h_k |_{p=2} = \left( \frac{p-1}{p} \right)^{p-2} k(1 + O(k^{-1})), \quad \text{as} \quad k \to \infty,
\]

which means that \( \sum_{k=1}^{\infty} R_k^{-1} = \infty \). Condition (22) is not satisfied, but according to the comment above (30) it is sufficient to show that conditions (30) and (31) hold instead of (22). Condition (31) follows from (28). To verify condition (30) we substitute \( \tau_k = 1 \) and \( h_k = k^{\frac{p-1}{p}} \) into (13) and using (28) we have

\[
H(k, v) = v + \frac{h_{k+1}}{h_k} G_k - \frac{h_{k+1}^p (v + G_k)}{\Phi(h_k^p + \Phi^{-1}(v + G_k))}
\]

\[
= v + \frac{h_{k+1}}{h_k} G_k - (v + G_k) \left( \frac{h_{k+1}}{h_k} \right)^p \left( 1 + \frac{\Phi^{-1}(v + G_k)}{k} \right)^{1-p}
\]

\[
= v + \left( 1 + \frac{1}{k} \right)^{\frac{p-1}{p}} G_k - \left( 1 + \frac{1}{k} \right)^{p-1} (v + G_k) \left( 1 + \frac{\Phi^{-1}(v + G_k)}{k} \right)^{1-p}
\]

\[
= v + \left( 1 + \frac{p-1}{pk} + o(k^{-1}) \right) G_k
\]

\[
- \left( 1 + \frac{p-1}{k} + o(k^{-1}) \right) (v + G_k) \left( 1 + \frac{(1-p)\Phi^{-1}(v + G_k)}{k} + o(k^{-1}) \right)
\]

\[
= \frac{p-1}{k} \frac{G_k}{p} - (v + G_k) + |v + G_k|^p + o(k^{-1}),
\]

as \( k \to \infty \). Denote by \( A \) the expression in brackets and let \( \alpha := (\frac{p-1}{p})^{p-1} \). Then

\[
G = \alpha \left( 1 - \frac{p-1}{2pk} + o(k^{-1}) \right)
\]

and

\[
A = \frac{\alpha}{p} \left( 1 - \frac{p-1}{2pk} + o(k^{-1}) \right)
\]

\[
+ \left[ v + \alpha \left( 1 - \frac{p-1}{2pk} + o(k^{-1}) \right) \right] \left[ \Phi^{-1} \left( v + \alpha - \frac{\alpha(p-1)}{2pk} + o(k^{-1}) \right) - 1 \right]
\]

\[
= \frac{\alpha}{p} - \frac{\alpha}{2pqk} + o(k^{-1})
\]

\[
+ \left[ v + \alpha - \frac{\alpha}{2qk} + o(k^{-1}) \right] \left[ \Phi^{-1}(v + \alpha) \Phi^{-1} \left( 1 - \frac{\alpha}{(\alpha + v)2qk} + o(k^{-1}) \right) - 1 \right]
\]

\[
= \frac{\alpha}{p} - \frac{\alpha}{2pqk} + o(k^{-1})
\]

\[
+ \left[ v + \alpha - \frac{\alpha}{2qk} + o(k^{-1}) \right] \left[ \Phi^{-1}(v + \alpha) \left( 1 - \frac{\alpha}{2(v + \alpha)pk} + o(k^{-1}) \right) - 1 \right]
\]
\[
= \frac{\alpha}{p} - \frac{\alpha}{2pqk} + o(k^{-1})
\]
\[
+ \left[ v + \alpha - \frac{\alpha}{2qk} + o(k^{-1}) \right] \left[ \Phi^{-1}(v + \alpha) - \frac{\alpha\Phi^{-1}(v + \alpha)}{2(v + \alpha)pk} - 1 + o(k^{-1}) \right]
\]
\[
= \frac{\alpha}{p} + [v + \alpha|^{q} - (v + \alpha)] + \frac{\beta}{k} + o(k^{-1}),
\]
where the constant \( \beta \) can be computed explicitly but its value is not important.

Consequently,
\[
H(k, v) = \frac{p - 1}{k} \left[ \frac{\alpha}{p} + |v + \alpha|^{q} - (v + \alpha) \right] + o(k^{-1}) \quad \text{as } k \to \infty.
\]
The term \( o(k^{-1}) \) is such that \( \sum_{k} o(k^{-1}) \) is convergent and by a direct computation we find that \( \frac{\alpha}{p} + |v + \alpha|^{q} - (v + \alpha) \geq 0 \) for \( v \in \mathbb{R} \) with equality if and only if \( v = 0 \). This means that
\[
\sum_{k} H(k, v) = \infty
\]
whenever \( v \neq 0 \).

It follows from (33) that there exists \( \varepsilon > 0 \) such that
\[
\liminf_{k \to \infty} \log k \sum_{j=k}^{\infty} d_j(j+1)^{p-1} > \frac{1}{2(1-\varepsilon)} \left( \frac{p-1}{p} \right)^{p-1} = \frac{1}{2q(1-\varepsilon)} \left( \frac{p-1}{p} \right)^{p-2},
\]
i.e.,
\[
\liminf_{k \to \infty} \log k \sum_{j=k}^{\infty} \left( \frac{p}{p-1} \right)^{p-2} \sum_{j=k}^{\infty} (1-\varepsilon)C_j > \frac{1}{4},
\]
where \( C \) is given by (20). Using the discrete l’Hospital rule (see, e.g., [1])
\[
\lim_{k \to \infty} \frac{\sum_{j=1}^{k-1} (1/j)}{\log k} = 1,
\]
and hence, in view of (16) and (34),
\[
\liminf_{k \to \infty} \left( \sum_{j=1}^{k-1} R_j^{-1} \right) \left( \sum_{j=k}^{\infty} (1-\varepsilon)C_j \right) > \frac{1}{4}.
\]
According to Lemma 3.1 it means that equation (23) is oscillatory, and we have consequently by Theorem 2.3 that also equation (29) is oscillatory.

4. Open problems and conjectures

In this concluding section we formulate some open problems related to the results of the previous section.

(i) As mentioned before, the half-linear Euler differential equation (26) has the principal solution \( h(t) = t^{\frac{1}{p-1}} \) and nonprincipal solutions behave asymptotically as \( x(t) = t^{\frac{1}{p-1}} \log^{2/p} t \), see [13]. In the discrete case, it is not clear what is the “right” Euler equation even in the linear case. In [1], the equation
\[
k(k+1)\Delta^2 x_k + ak\Delta x_k + bx_k = 0
\]
(35)
is referred to as the discrete second order Euler difference equation.

The reason for this terminology is that similarly to the continuous case, a solution of this equation is in the form $x_k = \frac{\Gamma(k+\lambda)}{\Gamma(k)}$, where $\lambda$ is a solution of the quadratic equation $\lambda(\lambda - 1) + a\lambda + b = 0$. However, equation (35) is not in the self-adjoint form, so the theory of Sturm-Liouville second order difference equations does not apply to (35). In the context of self-adjoint equations, as Euler equation is usually regarded the equation

$$\Delta^2 x_k + \frac{b}{k(k+1)} x_{k+1} = 0, \quad b \in \mathbb{R},$$

but in contrast to (35), solutions of (36) cannot be expressed by an explicit formula.

(ii) In our treatment of the half-linear Euler type difference equation we started “from the end”. Following the continuous case, we took the sequence $x_k = k^{\frac{p-1}{p}}$ and then we computed the sequence $\tilde{c}$ in (24) by formula (27). However, as mentioned in the previous section, we know that $x_k$ is the recessive solution of (24) only for $p \geq 2$. We conjecture that this is the case also for $p \in (1, 2)$. We also conjecture, again based on the continuous case, that dominant solutions of (24) behave asymptotically as

$$x_k = k^{\frac{p-1}{p}} \left( \sum_{j=1}^{k} \frac{1}{j} \right)^{2/p}.$$

(iii) Another open problem is related to the so-called half-linear Riemann-Weber differential equation

$$(\Phi(x'))' + \left[ \gamma \frac{\lambda}{p^2 \log^2 k} \right] \Phi(x) = 0$$

with $\gamma$ given in (26).

It is known that (37) is nonoscillatory if and only if $\lambda \leq \mu := \frac{1}{4} \left( \frac{p-1}{p} \right)^{p-1}$. We conjecture that the discrete version of (37) (with $\tilde{c}$ given by (27))

$$\Delta(\Phi(\Delta x_k)) + \left[ \frac{\lambda}{(k+1)^p \log^2 (k+1)} \right] \Phi(x_{k+1}) = 0$$

is nonoscillatory if and only if $\lambda \leq \mu$. This conjecture holds for $\lambda < \mu$ (see [5, Theorem 5.1] and if $p \geq 2$ is also for $\lambda > \mu$ (as a consequence of Theorem 3.2). Moreover, it is supported by the linear case $p = 2$, see [17], where, among others, perturbations of linear Euler type difference equations are investigated.

(iv) The last open problem concerns the results of the paper [19], where the pair of perturbed Euler type differential equations with different powers by “half-linearity” is investigated. Oscillatory properties of the equation

$$(\Phi_p(x'))' + \frac{\gamma}{p^2} \left[ 1 + q\delta(t) \right] \Phi_p(x) = 0,$$

where $\delta$ is a positive continuous function and $q$ is the conjugate number to $p$, are compared with the equation of the same form containing the power function $\Phi_p(x) := |x|^{p'-2}x$. In particular, it is proved that if $1 < p < p'$ and the equation with the power $p'$ is oscillatory then (38) has the same property. The subject of the present investigation is to extend these results to the perturbed half-linear difference equation (29).
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Principal and Nonprincipal Solutions in the Oscillation Theory of Half-Linear Differential Equations

ONDŘEJ DOŠLÝ and JANA ŘEZNÍČKOVÁ

Abstract. We discuss the role played by principal and nonprincipal solutions of the half-linear second order differential equation

\[(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad p > 1,\]

in the half-linear oscillation theory.

1. Introduction

In this paper we will study oscillatory properties of solutions of the half-linear second order differential equation

\[(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad p > 1,\]  

(1)

where \(r, c\) are continuous functions, \(r(t) > 0\). Our principal concern is to show what role play principal and nonprincipal solutions in the oscillation theory of (1).

The concept of the principal solution of the second order linear differential equation

\[(r(t)x')' + c(t)x = 0\]  

(2)

was introduced by Leighton and Morse [15] and basic properties of this solution were investigated by Hartman, see [12] for a basic survey. It was shown that nonoscillatory equation (2) possesses a unique (up to a nonzero multiplicative factor) solution \(h\), called the principal solution, with the property that

\[
\lim_{t \to \infty} \frac{h(t)}{x(t)} = 0 \quad \text{for any solution } x \text{ linearly independent of } h. 
\]

(3)

An equivalent characterization of the principal solution is

\[
\int_{\infty}^{\infty} \frac{dt}{r(t)h^2(t)} = \infty.
\]

(4)

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since this integral is convergent for any solution linearly independent of \( h \). Solutions linearly independent of the principal solution are called nonprincipal solutions.

To explain the role played by the principal and nonprincipal solutions in the oscillation theory, we treat the linear equation (2) as first. Together with this equation we consider the nonoscillatory equation

\[
(r(t)x')' + \tilde{c}(t)x = 0, \quad (5)
\]

where \( \tilde{c} \) is a continuous function. Denote by \( h \) and \( \tilde{x} \) principal and nonprincipal solutions of (5), respectively, for which

\[
r(\tilde{x}'h - \tilde{c}h') = 1 (\text{such solutions always exist}).
\]

The proofs of the following statements can be found in [2] and (sometimes implicitly) in [17].

**Theorem 1.1.** Equation (2) is oscillatory provided one of the following conditions holds:

(i) (Leighton-Wintner type criterion)

\[
\int_{t}^{\infty} (c(t) - \tilde{c}(t))h^2(t) \, dt = \infty; \quad (6)
\]

(ii) (Hille-Nehari type criterion with the principal solution) The integral in (6) is convergent and

\[
\liminf_{t \to \infty} \left( \int_{t}^{\infty} r^{-1}(s)h^{-2}(s) \, ds \right) \left( \int_{t}^{\infty} (c(s) - \tilde{c}(s))h^2(s) \, ds \right) > \frac{1}{4}; \quad (7)
\]

(iii) (Hille-Nehari type criterion with the nonprincipal solution)

\[
\liminf_{t \to \infty} \left( \int_{t}^{\infty} r^{-1}(s)\tilde{x}^{-2}(s) \, ds \right) \left( \int_{t}^{\infty} (c(s) - \tilde{c}(s))\tilde{x}^2(s) \, ds \right) > \frac{1}{4}. \quad (8)
\]

The nonoscillatory counterpart of Theorem 1.1 reads as follows.

**Theorem 1.2.** Equation (2) is nonoscillatory provided one of the following conditions holds:

(i) \( \limsup_{t \to \infty} \left( \int_{t}^{\infty} r^{-1}(s)h^{-2}(s) \, ds \right) \left( \int_{t}^{\infty} (c(s) - \tilde{c}(s))h^2(s) \, ds \right) < \frac{1}{4}; \)

\[
\liminf_{t \to \infty} \left( \int_{t}^{\infty} r^{-1}(s)h^{-2}(s) \, ds \right) \left( \int_{t}^{\infty} (c(s) - \tilde{c}(s))h^2(s) \, ds \right) > \frac{3}{4}; \quad (9)
\]

(ii) \( \limsup_{t \to \infty} \left( \int_{t}^{\infty} r^{-1}(s)\tilde{x}^{-2}(s) \, ds \right) \left( \int_{t}^{\infty} (c(s) - \tilde{c}(s))\tilde{x}^2(s) \, ds \right) < -\frac{1}{4}; \)

\[
\liminf_{t \to \infty} \left( \int_{t}^{\infty} r^{-1}(s)\tilde{x}^{-2}(s) \, ds \right) \left( \int_{t}^{\infty} (c(s) - \tilde{c}(s))\tilde{x}^2(s) \, ds \right) > -\frac{3}{4}. \quad (10)
\]
In the previous theorems, equation (2) is regarded as a perturbation of the nonoscillatory equation (5) and the previous theorems state, roughly speaking, that (2) becomes oscillatory (remains nonoscillatory) if the difference $c - \tilde{c}$ is sufficiently large (not too large) with respect to nonoscillation of (5). Note that the term $\int t^{r_{1}}h^{-2}$ in (7) (which tends to $\infty$ as $t \to \infty$) can be also expressed as $\tilde{x}(t)/h(t)$ and the term $\int_{t}^{\infty}r^{-1}\tilde{x}^{-2}$ (which tends to zero) as $h(t)/\tilde{x}(t)$. Rate of the convergence $h(t)/\tilde{x}(t) \to 0$ “measures” how much (5) is nonoscillatory. Consequently, from this point of view, in Hille-Nehari criteria from Theorems 1.1, 1.2, the first term in the product characterizes the “nonoscillation stability” of (5) and the second term characterizes the “size of perturbation”.

The idea of the proof of the criteria given in this section is the following. The transformation $x = hy$ resp. $x = \tilde{x}y$ transforms (2) into the equation

$$(R(t)y')' + C(t)y = 0,$$

where $R = rh^{2}, C = (c - \tilde{c})h^{2}$ resp. $R = r\tilde{x}^{2}, C = (c - \tilde{c})\tilde{x}^{2}$.

Now, (11) is regarded as a perturbation of the one-term equation $(R(t)y')' = 0$. The application of the classical methods of linear oscillation theory (variational principle, Riccati technique) then gives (non)oscillation criteria for (11). These results “transformed back” to (2) then yield criteria given in Theorems 1.1, 1.2. We refer to [2] for details.

2. HALF-LINEAR EQUATIONS

In this section we show how the previous linear results can be extended to (1). We start with the situation when (1) is regarded as a perturbation of the one term equation

$$(r(t)\Phi(x'))' = 0.$$  

We refer to [1, 13, 14] for the proofs of the next statements.

**Theorem 2.1.** Equation (1) is oscillatory provided one of the following conditions is satisfied:

(i) $\int_{t}^{\infty}r^{-q}(t)\ dt = \infty$ and $\int_{t}^{\infty}c(t)\ dt = \infty, \quad q := \frac{p}{p - 1}.$  

(ii) The first condition in (13) holds and

$$\liminf_{t \to \infty} \left(\int_{t}^{\infty}r^{1-q}(s)\ ds\right)^{p-1}\int_{t}^{\infty}c(s)\ ds > \frac{1}{p} \left(\frac{p - 1}{p}\right)^{p-1}.$$  

(iii) The integral

$$\int_{t}^{\infty}r^{1-q}(t)\ dt < \infty$$

and

$$\liminf_{t \to \infty} \left(\int_{t}^{\infty}r^{1-q}(s)\ ds\right)^{p-1}\int_{t}^{\infty}c(s)\ ds > \frac{1}{p} \left(\frac{p - 1}{p}\right)^{p-1}.$$
(iv) Condition (14) holds, \(c(t) \geq 0\) for large \(t\), and
\[
\liminf_{t \to \infty} \frac{1}{\int_t^\infty r^{1-q}(s) \, ds} \int_t^\infty c(s) \left( \int_s^\infty r^{1-q}(\tau) \, d\tau \right)^p \, ds > \left( \frac{p-1}{p} \right)^p.
\]

The nonoscillatory counterparts of some results of the previous theorem are presented in the next statement.

**Theorem 2.2.** Suppose that \(\int_1^\infty r^{1-q}(t) \, dt = \infty\) and \(\int_1^\infty c(t) \, dt\) converges. If
\[
\limsup_{t \to \infty} \left( \int_t^\infty r^{1-q}(s) \, ds \right)^{p-1} \left( \int_1^\infty c(s) \, ds \right) < \frac{1}{p} \left( \frac{p-1}{p} \right)^{p-1},
\]
\[
\liminf_{t \to \infty} \left( \int_t^\infty r^{1-q}(s) \, ds \right)^{p-1} \left( \int_1^\infty c(s) \, ds \right) > -\frac{2p-1}{p} \left( \frac{p-1}{p} \right)^{p-1},
\]
then (1) is nonoscillatory. If (14) holds, the statement of the theorem remains to hold provided \(\int_1^\infty r^{1-q}\) and \(\int_1^\infty c\) in (15), (16) are replaced by \(\int_1^\infty r^{1-q}\) and \(\int^t c\).

Now we recall the concept of the principal solution of nonoscillatory half-linear equation. A solution \(h\) of (1) is said to be principal (at \(\infty\)), if its logarithmic derivative is less than logarithmic derivative of any linearly independent solution \(x\) of (1) for large \(t\), i.e.,
\[
\frac{h'(t)}{h(t)} < \frac{x'(t)}{x(t)} \text{ for large } t.
\]

This also means that the principal solution of (1) defines via the formula \(w = r\Phi(h'/h)\) the minimal solution of the associated Riccati equation
\[
w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad q = \frac{p}{p-1}.
\]

Note that this definition, applied to linear equation (2), defines the same solution as characterized by (3) and (4), see [11, 16]. These two “linear” characterizations of the principal solution do not extend directly to half-linear equation since they are based on the linearity of the solution space and on the Wronskian identity which are lost in the half-linear case, see [6, Sec. 1.3].

In the remaining part of this section we regard (1) as a perturbation of the nonoscillatory general half-linear equation
\[
(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0.
\]
In contrast to the linear case, this equation cannot be reduced to a one-term equation of the form (12). Hence, similarly to the construction of the principal solution, linear methods have to be substantially modified. The statements presented below can be found (in a slightly modified form with respect to the presentation given here) in the papers [5, 7, 10].

We start with the Leighton-Wintner type criterion ([5]) which is the half-linear version of the part (i) of Theorem 1.1.
Theorem 2.3. Let $h$ be the positive principal solution of (17). If
\[ \int_{\infty}^{\infty} (c(t) - \tilde{c}(t))h^p(t)\,dt = \infty, \]
then (1) is oscillatory.

The next statement is the half-linear extension of Theorem 1.1 (ii) and Theorem 1.2, its proof can be also found in [5].

Theorem 2.4. Suppose that (17) is nonoscillatory and possesses the principal solution $h$ such that $h'(t) \neq 0$ for large $t$. Denote
\[ G(t) := r(t)h(t)\Phi(h'(t)), \quad R(t) := r(t)h^2(t)|h'(t)|^{p-2}. \]
Further suppose that
\[ \liminf_{t \to \infty} |G(t)| > 0, \quad \int_{t}^{\infty} R^{-1}(s)\,ds = \infty, \]
(i) If
\[ \liminf_{t \to \infty} \left( \int_{t}^{t} R^{-1}(s)\,ds \right) \int_{t}^{\infty} (c(s) - \tilde{c}(s))h^p(s)\,ds > \frac{1}{2q}, \]
then (1) is oscillatory.
(ii) If
\[ \limsup_{t \to \infty} \left( \int_{t}^{t} R^{-1}(s)\,ds \right) \int_{t}^{\infty} (c(s) - \tilde{c}(s))h^p(s)\,ds < \frac{1}{2q}, \]
\[ \liminf_{t \to \infty} \left( \int_{t}^{t} R^{-1}(s)\,ds \right) \int_{t}^{\infty} (c(s) - \tilde{c}(s))h^p(s)\,ds > -\frac{3}{2q}, \]
then (1) is nonoscillatory.

We finish this section with the next statement. Its first part is proved in [7], while the second statement is a result which will be submitted for publication in the next future.

Theorem 2.5. (i) Let $\tilde{x}$ be a positive solution of (17) such that $\tilde{x}'(t) > 0$ for large $t$, say $t > T$, $\int_{\infty}^{\infty} r^{-1}(t)\tilde{x}^{-2}(t)(\tilde{x}'(t))^2\,dt < \infty$, and denote
\[ \tilde{R}(t) := \int_{t}^{\infty} \frac{ds}{r(s)\tilde{x}^2(s)(\tilde{x}'(s))^{p-2}}. \]
Suppose that
\[ \lim_{t \to \infty} \tilde{R}(t)\tilde{G}(t) = \infty, \]
where the function $\tilde{G}$ is given by formula (18) with $\tilde{x}$ instead of $h$. If
\[ \limsup_{t \to \infty} \tilde{R}(t) \int_{T}^{t} (c(s) - \tilde{c}(s))\tilde{x}^p(s)\,ds < \frac{1}{2q}, \]
and
\[
\liminf_{t \to \infty} R(t) \int_{T}^{t} (c(s) - \tilde{c}(s)) \tilde{x}^p(s) \, ds > -\frac{3}{2q}
\]
for some \( T \in \mathbb{R} \) sufficiently large, then (1) is nonoscillatory.

(ii) Let \( h \) be the positive principal solution of (17) such that (19) holds. If \( c(t) \geq \tilde{c}(t) \) for large \( t \) and
\[
\liminf_{t \to \infty} \frac{1}{\int_{T}^{t} R^{-1}(s) \, ds} \int_{T}^{t} (c(s) - \tilde{c}(s)) h^p(s) \left( \int_{s}^{\infty} R^{-1}(\tau) \, d\tau \right)^2 \, ds > \frac{1}{2q}
\]
for \( T \) sufficiently large, then equation (1) is oscillatory.

3. An oscillation criterion

In this section we prove a new oscillation criterion for (1). If \( h \equiv 1 \) in this criterion, it reduces to the main result of [3].

**Theorem 3.1.** Let \( p \geq 2 \). Suppose that \( \tilde{c}(t) \leq c(t) \) for large \( t \), (17) is nonoscillatory and possesses the positive principal solution \( h \) such that
\[
\int_{t}^{\infty} r^{1-q}(t) h^{-q}(t) H(t, v) \, dt = \infty \tag{20}
\]
for every \( v > 0 \), where
\[
H(t, v) := |v + G(t)|^q - qv \Phi^{-1}(G(t)) - |G(t)|^q \tag{21}
\]
with \( G \) given by (18).

Finally, suppose that \( \int_{t}^{\infty} c^*(t) \, dt < \infty \), where \( c^* := (c - \tilde{c}) h^p \). If
\[
\int_{t}^{\infty} r^{1-q}(t) \left| w_h(t) + h^{-p}(t) \int_{t}^{\infty} c^*(s) \, ds \right|^{q-2} \times \exp \left\{ -p \int_{t}^{\infty} r^{1-q}(s) \Phi^{-1} \left( w_h(s) + h^{-p}(s) \int_{s}^{\infty} c^*(\tau) \, d\tau \right) \, ds \right\} \, dt < \infty, \tag{22}
\]
where \( w_h = r \Phi(h'/h) \), then (1) is oscillatory.

**Proof.** Suppose, by contradiction, that (1) is nonoscillatory and let \( \tilde{x} \) be its principal solution. Since \( p \geq 2 \), by [4]
\[
I(\tilde{x}) := \int_{t}^{\infty} \frac{dt}{r(t) \tilde{x}^2(t) |\tilde{x}'(t)|^{p-2}} = \infty. \tag{23}
\]
Put \( v = h^p(\tilde{w} - w_h) \), \( \tilde{w} = r \Phi(\tilde{x}'/\tilde{x}) \). Then the inequality \( \tilde{c} \leq c \) implies that \( v(t) \geq 0 \) for large \( t \), see [11], and \( v \) is a nonincreasing solution of the equation
\[
v' + (c(t) - \tilde{c}(t)) h^p(t) + (p-1) r^{1-q}(t) h^{-q}(t) H(t, v) = 0. \tag{24}
\]
Hence there exists the nonnegative limit \( v(\infty) \) and by (20) \( v(\infty) = 0 \). Indeed, from (24), for \( T < t \),
\[
v(t) + \int_T^t c^*(s) \, ds + (p - 1) \int_T^t r^{1-q}(s) h^{-q}(s) H(s, v(s)) \, ds = v(T),
\]
(25)

hence
\[
(p - 1) \int_T^t r^{1-q}(s) h^{-q}(s) H(s, v(s)) \, ds \leq v(T).
\]

Letting \( t \to \infty \) in (25) and replacing \( T \) by \( t \), we can write (24) in the integral form
\[
v(t) = \int_t^\infty c^*(s) \, ds + (p - 1) \int_t^\infty r^{1-q}(s) h^{-q}(s) H(s, v(s)) \, ds \geq \int_t^\infty c^*(s) \, ds.
\]

Hence,
\[
\tilde{w}(t) = h^{-p}(t)v(t) + w_h(t) \geq w_h(t) + h^{-p}(t) \int_t^\infty c^*(s) \, ds.
\]

Then, since \( \tilde{x}' = \frac{\Phi^{-1}(\tilde{x}(t))}{\Phi^{-1}(x(t))} \tilde{x} \), we have
\[
\tilde{x}(t) = \exp \left\{ \int_t^t r^{1-q}(s) \Phi^{-1}(w(s)) \, ds \right\}
\]
\[
\geq \exp \left\{ \int_t^t r^{1-q}(s) \Phi^{-1} \left( w_h(s) + h^{-p}(s) \int_s^\infty c^*(\tau) \, d\tau \right) \, ds \right\}
\]
and (we skip the argument \( t \))
\[
|\tilde{x}'|^p - 2 = \left| \frac{\tilde{w}^{1/(q-1)(p-2)}}{r} \right| |\tilde{x}'|^p - 2 = \left| \frac{\tilde{w}^{1-2/q}}{r} \right| |\tilde{x}'|^p - 2
\]
\[
\geq r^{q-2} \left| w_h + h^{-p} \int_t^\infty c^* \right|^{2-q} \exp \left\{ (p - 2) \int_t^t r^{1-q} \Phi^{-1} \left( w_h + h^{-p} \int_s^\infty c^* \right) \, ds \right\}.
\]

Consequently,
\[
\frac{1}{r \tilde{x}^2 |\tilde{x}'|^p - 2}
\]
\[
< \frac{1}{r^{q-1} |w_h + h^{-p} \int_t^\infty c^*|^{2-q}} \exp \left\{ -p \int_t^t r^{1-q} \Phi^{-1} \left( w_h + h^{-p} \int_s^\infty c^* \right) \, ds \right\}
\]
\[
= r^{1-q} \left| w_h + h^{-p} \int_t^\infty c^* \right|^{q-2} \exp \left\{ -p \int_t^t r^{1-q} \Phi^{-1} \left( w_h + h^{-p} \int_s^\infty c^* \right) \, ds \right\}.
\]

This implies, by (22) that
\[
\int_{\infty}^\infty \frac{dt}{r(t) \tilde{x}^2(t) |\tilde{x}'(t)|^p - 2} < \infty
\]
and this contradiction with (23) completes the proof. \( \Box \)
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RESEARCH STABILITY OF SOLUTIONS OF THE SYSTEM LINEAR DIFFERENTIAL EQUATIONS WITH RANDOM COEFFICIENTS

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Abstract. We find necessary and enough conditions for $L_2$-stability of solutions of the linear system differential equations with random (Semi-Markovian) coefficients.

INTRODUCTION

Let’s consider the system of the linear differential equations

$$\frac{dX(t)}{dt} = A(t, \xi(t))X(t)$$

where $\xi(t)$ – semi Markovian random process, which have value $\theta_1, \ldots, \theta_n$.

We assume:

1. Semi-Markovian random process determined with intensities $q_{sk}$ ($s, k = 1, 2, \ldots, n$).
2. Always assume boundaries of initial values $X(0)$.
3. In time $t_j$ semi-Markovian process and solutions of system equations (1) have jump which defined the equations

$$X(t_j + 0) = C_{ks}X(t_j - 0), \quad \det C_{ks} \neq 0.$$  

1. FIRST SECTION

Definition 1.1. A zero solution of the system (1) is named asymptotic of stability in middle, if for the arbitrary solution $X(t)$ of the system (1) a limit correlation is executed

$$\lim_{t \to \infty} M(t) = \lim_{t \to \infty} \langle X(t) \rangle = 0$$

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Moments equations for the system (1) have kind:

\[ M_k(t) = \psi_k(t)N_k(t)M_k(0) + \int_0^t \psi_k(t-\tau)N_k(t-\tau)V_k(\tau)d\tau, \quad (3) \]

\[ V_k(t) = \sum_{s=1}^n q_{ks}(t)C_{ks}N_s(t)M_s(t)M_s(0) + \int_0^t \sum_{s=1}^n q_{ks}(t-\tau)C_{ks}N_s(t-\tau)V_k(\tau)d\tau, \quad (4) \]

where \( M_k(t) \) - moments of first order.

**Definition 1.2.** A zero solution of the system (1) is named asymptotic of stability in middle quadratic, if at the arbitrary initial conditions executed limit correlation

\[ \lim_{t \to \infty} \langle \|X(t)\|^2 \rangle = 0, \quad (5) \]

\[ \|X(t)\|^2 = \sum_{k=1}^n |x_k|^2. \quad (6) \]

It is possible to prove, that asymptotic of stability in middle quadratic equivalent implementation of correlation

\[ \lim_{t \to \infty} D(t) = 0, \quad D(t) \equiv \langle X(t)X^*(t) \rangle. \quad (7) \]

For research of stability in middle quadratic it is possible to use the system of the equations for the matrix r.m. second order

\[ D_k(t) = \psi_k(t)N_k(t)D_k(0)N^*_k(t) + \int_0^t \psi_k(t-\tau)N_k(t-\tau)W_k(\tau)N^*_k(t-\tau)d\tau, \quad (8) \]

\[ W_k(t) = \sum_{s=1}^n q_{ks}(t)C_{ks}N_s(t)D_s(0)N^*_s(t)C^*_{ks} + \]

\[ + \int_0^t \sum_{s=1}^n q_{ks}(t-\tau)C_{ks}N_s(t-\tau)W_s(\tau)N^*_s(t-\tau)C^*_{ks}d\tau. \quad (9) \]

**Definition 1.3.** A zero solution of the system of the equations (1) is named \( L_2 \)-stability, if the integral convergence

\[ I = \int_0^\infty \|X(t)\|^2 dt. \quad (10) \]

Obviously, that zero solution of the system (1) of \( L_2 \)-stability then and only after, when the matrix integral convergence

\[ D = \int_0^\infty D(t)dt. \quad (11) \]

That to find the condition of \( L_2 \)-stability terms, will point some auxiliary results.
Lemma 1.4. For matrices of second partial moments

\[ D_k(t) = \int_0^t XX^* f_k(t, X) dX, \quad (k = 1, \ldots, n). \]

inequality executed \( D_k(t) \geq 0 \).

Proof. So far as \( f_k(t, X) \geq 0 \), at the arbitrary vector have inequality

\[ Y^* D_k(t) Y = \int_0^t (Y^* X)(X^* Y) f_k(t, X) dX = \int_0^t |X^* Y|^2 f_k(t, X) dX \geq 0. \]

That is was required to prove. \( \square \)

Lemma 1.5. For the system of the integral equations (8) unevenness are executed

\[ W_k(t) \geq 0. \]

Proof. Solution of the system of the equations for \( W_k(t) \) it is possible to find method of the successive approaching at, taking

\[
W_k^{(j+1)}(t) = \sum_{s=1}^n q_{ks}(t) C_{ks} N_s(t) D_s(0) N_s^*(t) C_{ks}^* \\
+ \int_0^t \sum_{s=1}^n q_{ks}(t-\tau) C_{ks} N_s(t-\tau) W_s^{(j)}(\tau) N_s^*(t-\tau) C_{ks}^* d\tau, \quad W_k^{(0)}(t) \equiv 0. \tag{12}
\]

From the system of the equation (9) swims out, that at \( W_k^{(j)}(t) \geq 0 \) inequality are executed, so far as. From the system of the equation (8) swims out, that inequality \( W_k(t) \geq 0 \) (\( k = 1, \ldots, n \)) are executed and-symmetric matrices.

That is was required to prove. \( \square \)

Theorem 1.6. Let’s assume that (1)–(3) conditions are executed. For that a zero solution of the system (1) was \( L_2 \)-stability, it is necessary, that were convergence unusual integrals

\[ I_k = \int_0^\infty \psi_k(t) N_k(t) D_k(0) N_k^*(t) dt < \infty. \]

Proof. From the system of the equation (8) swims out, that inequalities are executed

\[ D_k(t) \geq \psi_k(t) N_k(t) D_k(0) N_k^*(t), \]

integrating which, find inequalities

\[ D_k = \int_0^\infty D_k(t) dt \geq \int_0^\infty \psi_k(t) N_k(t) D_k(0) N_k^*(t) dt. \]

That is was required to prove. \( \square \)

It is possible to replace the conditions (9) by conditions of convergence of matrix unusual integrals

\[ I_{k0} = \int_0^\infty \psi_k(t) N_k(t) N_k^*(t) dt < \infty. \tag{13} \]
Indeed, for any symmetric matrix $D_k(0) \geq 0$ the number exists, such that. Thus inequalities are executed

$$I_k = \int_0^\infty \psi_k(t)N_k(t)D_k(0)N_k^*(t)dt \geq \int_0^\infty \psi_k(t)N_k(t)\rho_k EN_k^*(t)dt \geq \rho_k I_{k0}. \quad (14)$$

We are integrating system of the equations (8) on interval from 0 to infinity. Noting $D_k = \int_0^\infty D_k(t)dt, W_k = \int_0^\infty W_k(t)dt. \quad (15)$ We will obtain the system of the linear algebra equations for matrices $W_k, D_k$

$$D_k = \int_0^\infty \psi_k(t)D_k(0)N_k^*(t)dt + \int_0^\infty \psi_k(t)N_k(t)W_k(t)N_k^*(t)dt, \quad (16)$$

$$W_k = \sum_{s=1}^n \int_0^\infty q_{ks}(t)C_{ks}N_s(t)(D_s(0) + W_s)N_s^*(t)C_{ks}^* dt. \quad (17)$$

**Lemma 1.7.** If inequalities $I_{k0} > 0 \quad (10)$ are executed, from boundaries of matrices the boundaries of matrices swim out.

**Proof.** From condition $D_k(0) \geq 0$ and equation (13) the inequalities swim out

$$D_k \geq \int_0^\infty \psi_k(t)N_k(t)W_k(0)N_k^*(t)dt. \quad (18)$$

Will choose a scalar multiplier, such that inequalities were executed

$$\rho_k I_{k0} \geq D_k.$$ 

From the matrix inequality swims out

$$\int_0^\infty \psi_k(t)N_k(t)(\rho_k E - W_k)N_k^* dt \geq 0.$$

That is was required to prove. \qed

Using the lemmas we obtain the following result:

**Theorem 1.8.** Let unusual matrix integrals

$$I_{k0} = \int_0^\infty \psi_k(t)N_k(t)N_k^*(t)dt$$

convergence and are positive definite symmetric matrices, that are $I_{k0} > 0$. Then necessary and sufficient condition of $L_2$-stability solutions of the system of the equation (1) there is a boundary of symmetric matrices that is implementation of kind inequalities.

Will convert the system of the matrix equation (13). We used by denotation

$$B_k = D_k(0) + W_k.$$  

It is possible to write down the system of the equation (13) in kind

$$B_k = D(0) + \sum_{s=1}^n \int_0^\infty q_{ks}(t)C_{ks}N_s(t), \quad (19)$$
Will consider monotonous linear operators, set by formulas
\[ L_{k_s}B_s = \int_0^\infty q_{k_s}(t)C_{k_s}N_s(t)B_sN_s^*(t)C_{k_s}^*dt. \]  
(20)

Thus it is possible to write down the system of the equation (8) in the operator form:
\[ B = D(0) + LB, \]  
(21)

where
\[ B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}. \]  
(22)

Consider notion of efficiency for matrices \( B \) if, where
\[ \begin{bmatrix} L_{11} & L_{12} & \ldots & L_{1n} \\ L_{21} & L_{22} & \ldots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \ldots & L_{nn} \end{bmatrix}. \]  
(23)

Thus the operator will be monotonous, so far as from inequalities the \( B(1) \geq B(2) \) unevenness \( LB(1) \geq LB(2) \) swims out. From theorem 2 swims out, that the system of the equation (13) has a solution then and only after, when the method of the successive approaching coincides
\[ B_k^{(j+1)} = D_k(0) + \sum_{s=1}^n \int_0^\infty q_{k_s}(t)C_{k_s}N_s(t)B_s^{(j)}N_s^*(t)C_{k_s}^*dt. \]  
(24)

As an existence of limited matrices \( W_k \geq 0 \), which satisfy the system of the equations (13), pursuant to theorem 2 equivalent the solutions, we obtain the following result.

**Theorem 1.9.** Let for the system of the linear differential equation (1) with jump of solutions (3) conditions of theorem 1, 2 are executed.

For a zero solution of the system (1) was \( L_2 \)-stability it is necessary and enough, that one of equivalent conditions was executed:

1) The system of the equation (15) at any \( D_k(0) > 0 \) had a positive solution.
2) System of the equation (15) at \( D_k(0) = E \) had a positive solution.
3) The method of the successive approaching coincided (17).
4) The operator had a spectral radius, less from unit.

For the asymptotic \( L_2 \)-stability solutions of the system (1) enough, that at some matrices \( B_k > 0 \) inequality were executed
\[ B_k - \sum_{s=1}^n \int_0^\infty q_{k_s}(t)C_{k_s}N_s(t)B_sN_s^*(t)C_{k_s}^*dt > 0. \]
Example 1.10. Will find the condition of the $L_2$-stability solutions of the equation

$$\frac{dx(t)}{dt} = a(\xi(t))x(t), \quad a(\theta_k) \equiv a_k,$$

where $\xi(t)$ – Semi-Markovian process, that acquires two states with the set intensities

$$q_{12} = q_{21} = \begin{cases} 0, & t > T \\ 2T^{-2}(T - t), & t \in [0, T] \end{cases}.$$

We assume, that the solution of the equation (18) has jumps

$$x(t_j + 0) = cx(t_j - 0),$$

which take place simultaneously with jumps of solutions of the equation (18).

The system of the equation (14) collects a kind

$$b_1 = D_1(0) + c^2 e^{2a_2 T} - 1 - 2a_2 T^2 b_2,$$

$$b_2 = D_2(0) + c^2 e^{2a_1 T} - 1 - 2a_1 T^2 b_1.$$  

Condition of the $L_2$-stability of solutions is given by such inequalities:

$$c^4 e^{2a_1 T} - 1 - 2a_1 T e^{2a_2 T} - 1 - 2a_2 T^2 < 1.$$

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ON ESTIMATE OF THE NUMBER OF SOLUTIONS FOR QUASILINEAR 6-TH ORDER BOUNDARY VALUE PROBLEM

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Abstract. A special technique based on the analysis of oscillatory behavior of linear equations is applied to investigation of nonlinear boundary value problem of sixth order. A quasi-linear sixth order equation \( x^{(6)} = f(t, x) \) is studied. We get the estimation of the number of solutions to the boundary value problems of the type

\[
\begin{align*}
x^{(6)} &= a(t)x + g(t, x), \\
x(a) &= A, \\
x'(a) &= A_1, \\
x''(a) &= A_2, \\
x'''(a) &= A_3, \\
x(b) &= B, \\
x'(b) &= B_1,
\end{align*}
\]

where \( g \) is continuous together with the partial derivative \( g_x \) which is supposed to be positive, and \( a(t) \) is continuous positive valued in \( t \in (a; +\infty) \) function. We assume also that at least one solution to the problem under consideration exists.

Introduction

We employ the idea by A. Perov [1, ch. 15] who studied the multiplicity of solutions to two-point the second order nonlinear boundary value problems. His approach is based on comparison of the behavior of solutions of the equation at some given solution of the BVP and at infinity. The first behavior is described in terms of the linear equation of variations and the second behavior is a consequence of requirements on a function \( f \) in the right side. Using this idea and the technique of the angular function (see [7]) the multiplicity results were obtained. To apply this idea to the study of higher order equations one have to choose another methods since the angular function technique hardly can be applied in this case. As an alternative the theory of oscillation of linear equations of higher order can be used. Multiplicity results for the third order BVPs were obtained in [2] by combining the idea of A. Perov and some facts from the linear theory of conjugate points. The notion of a conjugate point is useful in our considerations. We use the definition of conjugate point by Kiguradze [5].

Definition 0.1. By \( m \)-th conjugate point \( \eta_m \) of \( a \) is called the minimal value of \( t = b, \ (b > a) \) where \( b \) is a \((m+n-1)\)-th zero (counting multiplicities in \([a; +\infty)\)) of solution of \( n \)-th order equation which has a zero at \( t = a \).

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As a by-product, a technique was elaborated for treatment of the higher dimensional case. In [3], $n$-th order BVPs were considered by straightforward generalizations of the results of [2]. However, only the case of $n-1$ (of total number $n$) boundary conditions prescribed at one end of the interval was treated.

We refer to the book by Kiguradze and Chanturia [5] with respect to two termed equations $y^{(n)} = p(t)y$, and to the article by Hunt [4] for sixth order equation.

In this paper we study the boundary value problem

$$x^{(6)} = f(t, x), \quad \text{where } f = a(t)x + g(t, x), \quad (1)$$

$$x(a) = A, \quad x'(a) = A_1, \quad x''(a) = A_2, \quad x'''(a) = A_3, \quad (2)$$

$$x(b) = B, \quad x'(b) = B_1, \quad (3)$$

under the assumption that there exists a particular solution $\xi(t)$ of the above problem and functions $a(t), g(t, x), g_x = \frac{\partial g}{\partial x}$ are continuous, $a(t) > 0$ for $t \in [a, +\infty)$ and $g_x(t, x) > 0$ for $(t, x) \in [a, +\infty) \times \mathbb{R}$. A solution of (1) can be described in terms of oscillatory behavior of the corresponding linear equation of variations

$$y^{(6)} = a(t)y + g_x(t, \xi(t))y, \quad (4)$$

which will play a significant role in our considerations. Our results are based on the theory of 6-th order differential equations of the form

$$y^{(6)} = p(t)y, \quad (5)$$

with boundary conditions (2)—(3). We assume that $p(t)$ is continuous and $p(t) > 0$.

Our principal result consists of estimation from below of the number of solutions to the boundary value problem (1)—(3) provided that it has at least one solution $\xi(t)$. This estimate depends on the oscillatory behavior of the equation (4).

The paper is organized as follows. In the subsequent section 1 we investigate linear equation of the form (5).

The main theorem of this section describes the two-parametric set of solutions to the equation (5), subject to initial data

$$y(a) = y'(a) = y''(a) = y'''(a) = 0. \quad (6)$$

In section 2 the multiplicity result is proved. The nonlinear problem (1)—(3) is considered, provided that $g$ is bounded. Our method of proof differs from that used by Perov [1] and is based on representation of the nonlinear equation (1) as a family of linear equations, the coefficients of which depend on solutions of (1) satisfying the initial value conditions (2).

1. Linear equation

In this section we consider the equation (5)

$$y^{(6)} = p(t)y,$$

with continuous coefficient $p > 0$. 

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**Definition 1.1.** Let us call a solution $x(t)$ of (5) $(l, 6-l)$-solution if there exist points $t_1, t_2$ ($a \leq t_1 < t_2$) such that this solution satisfies conditions

$$x^{(i)}(t_1) = 0, \quad (i = 0, \ldots, l-1),$$

$$x^{(i)}(t_2) = 0, \quad (i = 0, \ldots, 5-l).$$

Lemmas similar to Lemma 2.1 and Lemma 2.2 in [6] are valid for equation (5).

**Lemma 1.2.** If $y(t)$ is a solution of (5) and the values of $y^{(i)}$, $i = 0, \ldots, 5$ are non-negative (but not all zero) for $t = a$, then the functions $y^{(i)}(t)$, $i = 0, \ldots, 5$ are positive for $t > a$.

Proof. Consider the case of $y'(a) > 0$. First of all we show that $y'(t) > 0$, $t > a$. From statement of lemma we get that all other derivatives are nonnegative in $t = a$, therefore there exists an interval $[a, t_0]$ where $y'(t) > 0$ for any $t \in [a, t_0)$, $y(t) > 0$ for any $t \in (a, t_0)$, and $y'(t_0) = 0$.

The right hand side of

$$y'(t) = y'(a) + \sum_{k=2}^{5} y^{(k)}(a) \frac{(t-a)^{k-1}}{(k-1)!} + \int_{a}^{t} \int_{a}^{z} \int_{a}^{v} \int_{a}^{u} \int_{a}^{r} p(s)y(s) \, ds$$

is positive and grow up together with $t$ until $y(t)$ remain positive.

We have that $y'(t) \geq y'(a) > 0$ in $(a, t_0)$ and therefore $y'(t_0) > 0$, a contradiction.

In other cases, if some of the values $y^{(i)}(a)$ ($i = 0, 2, 3, 4, 5$) is positive, the proof is similar.

**Lemma 1.3.** If $u(t), v(t)$ are two different solutions of (5) and $u^{(i)}(a) \geq v^{(i)}(a)$, $i = 0, \ldots, 5$ (at least one inequality is strict) then $u^{(i)}(t) > v^{(i)}(t)$, $i = 0, \ldots, 5$ for $t > a$.

Proof. The result follows from Lemma 1.2, if we consider the function $w(t) = u(t) - v(t)$. This function is a solution of (5) and $w^{(i)}$, $i = 0, \ldots, 5$ are non-negative (but not all zero).

Let us state result by Kiguradze and Chanturia [5, Lemma 1.10, p. 26] adapted to the case of the 6-th order equation.

**Lemma 1.4.** The $k$-th conjugate point $\eta_k$ ($k = 1, 2, 3, \ldots$) of equation (5), ($p(t) > 0$), is represented by either $(4, 2)$-solution or $(2, 4)$-solution which has exactly $k-1$ zeros of odd order in interval $(a, \eta_k)$.

Now we can prove two important results.

**Lemma 1.5.** The $k$-th conjugate point $\eta_k$ of equation (5) is represented by either $(4, 2)$-solution or $(2, 4)$-solution, which has exactly $k-1$ simple zeros in interval $(a, \eta_k)$.
Proof. Define the function $\Phi_+(t)$ as

$$
\Phi_+(t) = x''x''' - x'x^{(4)} + xx^{(5)}.
$$

(8)

The function $\Phi_+(t)$ for the $(4,2)$-solution of equation (5) is nondecreasing positive valued function for any $t > 0$, since $\Phi_+(a) = 0$,

$$
\Phi_+(t) = \{(x^{(3)}y + x''x^{(4)}) - (x''x^{(4)} + x'x^{(5)}) + (x'x^{(5)} + xx^{(6)}) = x''y + p(t)x^2 \geq 0
$$

for any $t > 0$, and there exists an interval $(a, t^*)$ where $\Phi_+(t) \neq 0$.

It means that after quadruple zero there are zeros of order no greater than two, but from Lemma 1.4 we know that all zeros of $(4,2)$-solution which related to $\eta_k$ in interval $(a, \eta_k)$ are odd, therefore simple. □

**Lemma 1.6.** The k-th conjugate point $\eta_k$ continuously depends on the coefficient $p(t) > 0$ of the equation (5).

Proof. Consider equation (5) together with the initial conditions $y(a) = y'(a) = y''(a) = y'''(a) = 0$, $y^{(4)}(a) = \alpha$, $y^{(5)}(a) = \beta$.

(9)

Due to linearity of the equation we can consider the initial angle

$$
\Theta = \arctan \frac{\beta}{\alpha} \in (-\frac{\pi}{2}, 0).
$$

(10)

Let $\Theta_k$ be the angle corresponding to a solution $y_k(t)$ with the conjugate point $\eta_k$ which is a double zero, unstable under perturbations of the coefficient $p(t)$. However, by Lemma 1.2 a solution with $\Theta < \Theta_k$ satisfies $y(t) < y_k(t)$ for $t > 0$. If $\Theta$ is close enough to $\Theta_k$ the respective $y(t)$ must have two simple zeros on opposite sides of $\eta_k$ and close to it. These simple zeros change continuously together with $p(t)$. Since $\eta_k$ lies between, it changes continuously too. □

It can be shown (like in the work [7] for fourth order linear two-termed equations) that both sequences $\{\eta_i\}$ and $\{\Theta_i\}$ are ordered as $-rac{\pi}{2} < \Theta_2 < \ldots < \Theta_{2i} < \ldots < \Theta_{2i+1} < \ldots < \Theta_1 < 0$

and

$$
0 < \eta_1 < \eta_2 < \ldots .
$$

Evidently both monotone sequences $\{\Theta_{2i-1}\}$ and $\{\Theta_{2i}\}$ have limits, say, $\Phi_{odd}$ and $\Phi_{even}$ respectively. Computations show that in the case of $p(t) = 1$ one has that $\Phi_{odd} = \Phi_{even} = -\frac{\pi}{4}$.

2. Nonlinear equation

In this section, results for the nonlinear boundary value problem (1)—(3) are stated and proved.

Suppose $\xi(t)$ is a solution of the boundary value problem (1)—(3). Let $x(t, \alpha, \beta)$ be a solution of the equation (1), subject to the initial conditions (2) and

$$
x^{(4)}(a) - \xi^{(4)}(a) = r \cos \Theta, \quad x^{(5)}(a) - \xi^{(5)}(a) = r \sin \Theta.
$$

(11)
We denote \( z(t) = x(t) - \xi(t) \). Consider auxiliary linear equations
\[
\frac{d^6}{dt^6} z = \varphi(t, r, \Theta) z, \tag{12}
\]
where coefficient \( \varphi \) depends on \( x(t, r, \Theta) \) and \( \xi(t) \) as follows:
\[
\varphi = \frac{f(t, x(t, r, \Theta)) - f(t, \xi(t))}{x(t, r, \Theta) - \xi(t)}.
\]

We assume that at the points where denominators vanish the right hand sides are substituted by appropriate value of \( f_x \).

Our further considerations are based on the following observation.

**Lemma 2.1.** A solution \( x(t) \) of the initial value problem (1), (2), (11) satisfies also the conditions (3) (i.e. it is a solution of the problem (1)–(3)) if and only if there exists an extremal function \( Z(t) \) of the linear equation (12), which represents a conjugate to \( t = a \) point at \( t = b \), and such that
\[
\arctan \frac{z^{(4)}(a)}{z^{(5)}(a)} = \Theta. \tag{13}
\]

**Proof.** Let \( x(t) \) be a solution \((x(t) \neq \xi(t))\) of the problem (1)–(3). Necessity then follows readily from the observation that \( x - \xi \) is an extremal function, since it satisfies the equation (12) and has a quadruple zero at \( t = a \) and a double zero at \( t = b \).

Consider now linear equation (12) corresponding to some solution \( x(t) \) of (1), (2), (11). Suppose that \( Z(t) \) is an extremal function for (12) with a conjugate point at \( t = b \).

Without loss of generality we may assume that
\[
Z^{(4)}(a) = x^{(4)}(a) - \xi^{(4)}(a), \quad Z^{(5)}(a) = x^{(5)}(a) - \xi^{(5)}(a). \tag{14}
\]

Otherwise \( Z(t) \) should be multiplied by an appropriate constant. Both functions \( Z(t) \) and \( x(t) - \xi(t) \) are solutions to the same Cauchy problem for linear equation (12). Thus they are identical, and \( x(t) = Z(t) + \xi(t) \) satisfies also the boundary condition (3) at the right end of interval \((a, b)\). Hence the proof. \( \square \)

**Lemma 2.2.** Let \( g \) in (1) be bounded. Let \( \xi(t) \) be a solution of the problem (1)–(3) and \( x(t) \) be a solution of the initial value problem (1)–(2). Then the difference \( x(t) - \xi(t) \) cannot have more than \( m + 5 \) zeros (counting multiplicities) in \([a, b]\) for large enough \( r^2 = (x^{(4)}(a) - \xi^{(4)}(a))^2 + (x^{(5)}(a) - \xi^{(5)}(a))^2 \), if ordinary differential equation \( y^{(6)} = a(t)y \) has \( m \) conjugate points in \((a, b)\), and \( t = b \) is not a conjugate point.

**Proof.** We have that
\[
(x - \xi)^{(6)} = a(t)(x - \xi) + g(t, x) - g(t, \xi),
\]
\[
\frac{(x - \xi)^{(6)}}{r^2} = a(t)\frac{x - \xi}{r^2} + \frac{g(t, x) - g(t, \xi)}{r^2},
\]
therefore functions $u = \frac{x-k}{r}$ satisfy
\begin{align*}
u^{(6)} &= a(t)u + \delta, \\
u(a) &= u'(a) = u''(a) = u'''(a) = 0, \\
u^{(4)}(a) &= \alpha, \quad u^{(5)}(a) = \beta, \quad \alpha^2 + \beta^2 = r^2, \quad (15)\end{align*}
where $\delta \to 0$ as $r^2 \to +\infty$.

The function $u(t)$ tends to a solution $y(t)$ of the problem
\begin{align*}
y^{(6)} &= a(t)y, \\
y(a) &= y'(a) = y''(a) = y'''(a) = 0, \\
y^{(4)}(a) &= \alpha, \quad y^{(5)}(a) = \beta, \quad \alpha^2 + \beta^2 = r^2.\end{align*}

We have assumed that equation $y^{(6)} = a(t)y$ has exactly $m$ conjugate points in $(a;b)$ and $t = b$ is not a conjugate point. Then $y(t,\alpha,\beta)$ for any $(\alpha,\beta)$ has not more than $m + 5$ zeros, counting multiplicities in $[a;b]$. The same behavior have solutions of (15).

The proof is complete. \qed

**Theorem 2.3.** Let $\xi(t)$ be a solution of the problem (1)–(3). Suppose that function $y$ in (1) is bounded and there exists a solution of the problem (1)–(3) such that the interval $(a,b)$ contains exactly $k$ conjugate points $\eta_1, \eta_2, \ldots, \eta_k$ (to $t = a$) with respect to the equation of variations (4). Then the boundary value problem (1)–(3) has at least $2[k - m] + 1$ solutions ($\xi$ counted), if ordinary differential equation $y^{(6)} = a(t)y$ has $m$ conjugate points in $(a,b)$.

**Proof.** Fix $\Theta \in (-\pi/2,0)$ and consider a one parametric family of linear equations (12). For small $r$, solutions of (12) behave like ones of the equation of variations. Hence, there exist $k$ conjugate points $\eta_1, \eta_2, \ldots, \eta_k$ in the interval $(a,b)$ for small $r$. On the other hand for large $r$, this interval contains $m$ points $\eta_{\nu}$. The only way for conjugate point to leave the interval $(a,b)$, as $r$ varies from zero to infinity, is to pass over $t = b$. Let $S_k(\Theta) = \max|r: \eta_k = b|$. For any $\Theta \in [-\pi/2,0)$ and any $k$ such $S_k$ exists and form continuous one-parametric (dependent on $\Theta$) curve. Denote by $w_k(\Theta)$ the angle defining the first extremal function of the equation
\begin{equation*}z^{(6)} = \varphi(t, S_k(\Theta), \Theta)z,\end{equation*}
and consider the difference $w_k(\Theta) - \Theta$.

Note that $w_1(0) - 0$ is positive and $w_1(-\pi/2) = -\pi/2$ is negative. Hence, there exist $\Theta_k \in [-\pi/2,0)$ such that $w_k(\Theta_k) = \Theta_k$. By Lemma 2.1, the $k$-th extremal function of the equation (12) where $r = S_k(\Theta_k), \Theta = \Theta_k$, generates a solution to the boundary value problem (1)–(3). It’s true for any $k$. Next $|k - m|$ solutions to the boundary value problem (1)–(3) we get after consideration $\Theta \in (\pi/2,\pi)$. \qed

**Remark 2.4.** If we suppose that function $f(t,x) = g(t,x)$ (that is, $a(t) \equiv 0$) in (1). Where $g$ is increasing bounded function and there exists a solution of the problem (1)–(3) such that the interval $(a,b)$ contains exactly $k$ conjugate points...
η₁, η₂, . . . , ηₖ (to t = a) with respect to the equation of variation (4). Then the boundary value problem (1)-(3) has at least 2⁵ₖ + 1 solutions (ξ counted), because ordinary differential equation y⁽ⁿ⁾ = 0 has not conjugate points in (a, b), (that is, m = 0.)

**Example 2.5.** Let us show the example of using the results of Theorem 2.3. We consider the quasilinear boundary value problem

\[ x⁽ⁿ⁾ = 8^6 x + \arctan(8^7 x), \]
\[ x(0) = x'(0) = x''(0) = x'''(0) = x(1) = x'(1) = 0. \] (16)

By linearization we have the linear equation in variation with constant coefficient

\[ y⁽ⁿ⁾ = (8^6 + 8^7) y, \]

which has two extremal solutions with conjugate points (double zeros) in (0; 1), approximately, η₁ ≈ 0.58 and η₂ ≈ 0.89; but for large enough \( \rho \) we have equation with constant coefficient 8³ which has exactly one extremal solution with conjugate point in (0; 1) at the point \( t \approx 0.838. \)

By Theorem 2.3 the problem (16) has at least \((2 - 1)² = 2\) solution (ξ doesn’t counted).

We have computed solutions of nonlinear BVP (16). They are given on the Figure 1. (Second solution is given by the variable change \( x = -x. \)) A third one is a trivial solution \( \xi \equiv 0. \)

![Figure 1](image)

**Figure 1.** The three solutions of nonlinear boundary value problem (16): \( x₁ \) satisfies \( x₁⁽⁴⁾(0) = 0.00335685, x₁⁽⁵⁾(0) = -0.0393, x₂ \) satisfies \( x₂⁽⁴⁾(0) = -0.00335685, x₂⁽⁵⁾(0) = 0.0393, \) \( x₃ \equiv 0. \)

**Example 2.6.** Let us show the example of using the result of Remark 2.4 to Theorem 2.3. We consider the nonlinear boundary value problem

\[ x⁽ⁿ⁾ = \arctan(8^7 x), \]
\[ x(0) = x'(0) = x''(0) = x'''(0) = x(1) = x'(1) = 0. \] (17)

By linearization we have the linear equation of variations

\[ y⁽ⁿ⁾ = 8^7 y, \]
which has exactly two extremal solutions with conjugate point (double zero) in $(0, 1)$. By computation $\eta_1 \approx 0.592876$, $\eta_2 \approx 0.907561$. We have computed five $(2k + 1 = 2 \cdot 2 + 1 = 5)$ solutions of nonlinear boundary value problem. They are given on the Figures 2 and 3. Fifth solution is a trivial solution $\xi \equiv 0$.

![Figure 2](image1.png)  \(x_1 \) satisfies \(x_1^{(4)}(0) = 0.040216, x_1^{(5)}(0) = -0.307, \)

\(x_2 \) satisfies \(x_2^{(4)}(0) = -0.040216, x_2^{(5)}(0) = 0.307.\)

![Figure 3](image2.png)  \(x_3 \) satisfies \(x_3^{(4)}(0) = 0.0017779, x_3^{(5)}(0) = -0.0205, \)

\(x_4 \) satisfies \(x_4^{(4)}(0) = -0.0017779, x_4^{(5)}(0) = 0.0205, x_5 \equiv 0.\)

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ON POSITIVITY OF THE THREE TERM 2n-ORDER DIFFERENCE OPERATORS

P. HASIL

Abstract. We consider the symmetric, three-term, 2n-order difference equation
\[ a_k y_k + b_{k+n} y_{k+n} + a_{k+2n} y_{k+2n} = 0 \]
and we show that this equation possesses a positive solution for \( k \in \mathbb{Z} \) if and only if
the infinite symmetric matrix associated with this equation is nonnegative definite.

1. Introduction

Let \( a_k < 0, b_k > 0, k \in \mathbb{Z} \), be real-valued sequences and consider the symmetric
2n-order recurrence relation
\[ (\tau y)_k := a_k y_k + b_{k+n} y_{k+n} + a_{k+2n} y_{k+2n}. \]
We associate with the difference equation
\[ \tau y = 0 \tag{1} \]
the operator \( T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) with the domain
\[ \ell^2_0(\mathbb{Z}) = \{ y = \{ y_k \}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}), \text{ only finitely many } y_k \neq 0 \} \]
given by the formula \( T f = \tau f, f \in \ell^2_0(\mathbb{Z}) \), and we show that (1) has a positive
solution \( y_k, k \in \mathbb{Z} \), if and only if the operator \( T \) is nonnegative, i.e.
\[ \langle Ty, y \rangle \geq 0 \quad \forall y \in \ell^2_0(\mathbb{Z}), \]
where \( \langle \cdot, \cdot \rangle \) denotes the usual inner product in \( \ell^2(\mathbb{Z}) \). The statements presented in
this paper extend some results given in [1] where the case \( n = 1 \) is considered.

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2. Preliminary Results

Let $T$ be infinite symmetric matrix associated in the natural way with the operator $T$ and let us denote for $\mu \leq \nu, \mu, \nu \in \mathbb{Z}$ the truncations of $T$ by

$$t_{\mu,\nu} := \begin{pmatrix} b_\mu & 0 & \cdots & 0 & a_\mu & \cdots & 0 \\ 0 & b_{\mu+1} & 0 & \cdots & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & \ddots & \ddots & \ddots \\ a_\mu & 0 & \cdots & 0 & b_{\mu+n} & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots \\ \cdots & 0 & a_{\nu-n} & 0 & \cdots & 0 & b_\nu \end{pmatrix} \in \text{Mat}_{(\nu-\mu+1) \times (\nu-\mu+1)},$$

and their determinants by

$$d_{\mu,\nu} := \det(t_{\mu,\nu}).$$

For $r, s \in \mathbb{Z}, r \equiv s \pmod{n}$, we introduce the diagonal minors $D_{r,s}$, which are determinants of the submatrix of $t_{\mu,\nu}$ consisting of rows and columns which contain diagonal elements $b_r, b_{r+n}, b_{r+2n}, \ldots, b_{s-n}, b_s$.

The following statement gives us the relation between determinant $d_{\mu,\nu}$ and its minors $D_{r,s}$.

**Lemma 2.1.** It holds that

$$\prod_{i=0}^{n-1} D_{\mu+i,s_i} = d_{\mu,\nu} \prod_{i=0}^{n-1} D_{r_i,\nu-1-i},$$

where, for $i = 0, \ldots, n-1$,

$$s_i = \max \{ x \in \mathbb{Z} : x \equiv \mu + i \pmod{n}, x \leq \nu \},$$

$$r_i = \min \{ x \in \mathbb{Z} : x \equiv \nu - i \pmod{n}, x \geq \mu \}.$$

**Proof.** Without the change of the determinant, the matrix $t_{\mu,\nu}$ can be transformed into the block diagonal matrix $\tilde{t}_{\mu,\nu}$ with blocks $D_{\mu+i,s_i}$ or $D_{r_i,\nu-1-i}$, $i = 0, \ldots, n-1$. 

**Corollary 2.1.** It holds that

$$\frac{D_{r,\nu}}{D_{r,\nu-n}} \cdot d_{\mu,\nu-1} = d_{\mu,\nu} = \frac{D_{\mu,\nu}}{D_{\mu+n,s}} \cdot d_{\mu+1,\nu},$$

where $r := \min \{ x \in \mathbb{Z} : x \equiv \nu \pmod{n}, x \geq \mu \}$ and $s := \max \{ x \in \mathbb{Z} : x \equiv \mu \pmod{n}, x \leq \nu \}$. 
**Corollary 2.2.** It holds that

\[ D_{\mu,s_0} = b_\mu \cdot D_{\mu+n,s_0} - a_\mu^2 \cdot D_{\mu+2n,s_0}, \quad D_{r_0,v} = b_v \cdot D_{r_0,v-n} - a_v^2 \cdot D_{r_0,v-2n}. \]

**Proof.** Computing \( d_{\mu,v} \) by expanding it along its first row we obtain

\[ d_{\mu,v} = b_\mu \cdot d_{\mu+1,v} - a_\mu^2 \cdot \left( \prod_{i=1}^{n-1} D_{\mu+i,s_i} \right) \cdot D_{\mu+2n,s_0}. \]

Next, using Lemma 2.1, we have

\[ \prod_{i=0}^{n-1} D_{\mu+i,s_i} = b_\mu \cdot \left( \prod_{i=1}^{n-1} D_{\mu+i,s_i} \right) \cdot D_{\mu+n,s_0} - a_\mu^2 \cdot \left( \prod_{i=1}^{n-1} D_{\mu+i,s_i} \right) \cdot D_{\mu+2n,s_0}, \]

which is the first equality. The second equality can be proved similarly. \( \square \)

**Lemma 2.2.** Let \( \nu > \mu + n \) and suppose that \( t_{\mu,v} \geq 0 \) (i.e., the matrix \( t_{\mu,v} \) is nonnegative definite), then \( d_{\mu,v-n} > 0, \ldots, d_{\mu,v-1} > 0. \)

**Proof.** We prove this statement for \( d_{\mu,v-n} \). The rest can be proved similarly.

The assumption \( t_{\mu,v} \geq 0 \) implies

(i) \( d_{\mu,k} \geq 0, \forall k \in [\mu, \nu] \cap \mathbb{Z} \),
(ii) \( \Delta_{\mu+1,s_i} \geq 0, \forall k_i \in [0, \nu - \mu - i] \cap \mathbb{Z} \), where

\[ s_{k,i} = \max \{ x \in \mathbb{Z} : x \equiv \mu + i \pmod{n}, x \leq \nu - k_i \}. \]

Denote

\[ \ell := \min \{ x \in [\mu, \nu] \cap \mathbb{Z} : x \equiv \nu \pmod{n}, d_{\mu,x} = 0 \}. \]

Because \( d_{\mu,\mu} = b_\mu > 0, \ldots, d_{\mu,n+\mu} = \prod_{j=0}^{n-1} b_{\mu+j} > 0, \) we have \( \ell \geq \mu + n \).

Now suppose by contradiction that \( d_{\mu,v-n} = 0 \), which implies that \( \ell \leq \nu - n \) and compute

\[ d_{\mu,\ell+n} = b_{\ell+n} \cdot d_{\mu,\ell+n-1} - a_\mu^2 \cdot \left( \prod_{i=1}^{n-1} D_{\ell+i,\ell+1-i} \right) \cdot D_{\ell,n}, \]

\[ D_{r_0,\ell+n} = b_{\ell+n} \cdot D_{r_0,\ell} - a_\ell^2 \cdot D_{r_0,\ell-n}, \]

Using Corollary 2.1 we have \( 0 = d_{\mu,\ell} = D_{r_0,\ell} \cdot \frac{d_{\mu,\ell-1}}{D_{r_0,\ell-1}} \) which implies that \( D_{r_0,\ell} = 0 \) and using Lemma 2.1 we have \( 0 < d_{\mu,\ell-n} = K \cdot D_{r_0,\ell-n}, \) where \( K > 0, \) which implies that \( D_{r_0,\ell-n} > 0. \) Altogether, we obtained that \( D_{r_0,\ell+n} < 0, \) which contradicts our assumption \( t_{\mu,v} \geq 0. \) \( \square \)

Following statement can be proved easily using Lemma 2.2 (see [1, p. 74–75]).

**Lemma 2.3.** Operator \( T \) is nonnegative definite if and only if \( d_{\mu,v} > 0, \) for all \( \mu, v \in \mathbb{Z}, \mu \leq v. \)
Now, we present two auxiliary statements, which give us the possibility to construct a solution of the difference equation (1).

**Lemma 2.4.** Let $T \geq 0$, and $r, s$ be the same as in Corollary 2.1. Then

(i) $\frac{1}{b_\mu} < \frac{D_{\mu+n,s}}{D_{\mu,s}} < \frac{b_{\mu-n}}{a_{\mu-n}^2}$;

(ii) $\frac{1}{b_\nu} < \frac{D_{r,\nu-n}}{D_{r,\nu}} < \frac{b_{\nu+n}}{a_{\nu}^2}$;

(iii) The sequence $c_i := \left\{ \frac{D_{\mu+n,s+i}}{D_{\mu,s+i}} \right\}$ is increasing for $i \in \mathbb{N}_0$.

(iv) The sequence $d_i := \left\{ \frac{D_{r-in,\nu-n}}{D_{r-in,\nu}} \right\}$ is increasing for $i \in \mathbb{N}_0$.

**Proof.** We prove parts (i) and (iii). Parts (ii) and (iv) can be proved similarly.

To prove the first part, we expand $d_{\mu,\nu}$ along its first row and using Lemma 2.1 we obtain

$D_{\mu,s_0} = b_\mu \cdot D_{\mu+n,s_0} - a_{\mu}^2 \cdot D_{\mu+2n,s_0}$.

Because $a_{\mu}^2 \cdot D_{\mu+2n,s_0} > 0$, the left inequality in (i) holds.

Now, we expand the determinant $d_{\mu-n,\nu}$ and we obtain

$0 < D_{\mu-n,s_0} = b_{\mu-n} \cdot D_{\mu,s_0} - a_{\mu-n}^2 \cdot D_{\mu+n,s_0}$

which proves the right inequality in (i).

The part (iii) we prove by induction. Directly one can verify that

$\frac{D_{\mu+n,\mu+n}}{D_{\mu+n}} < \frac{D_{\mu+n,\mu+2n}}{D_{\mu+2n}}$.

Now, we multiply equalities

$D_{\mu,s} = b_\mu \cdot D_{\mu+n,s} - a_\mu^2 \cdot D_{\mu+2n,s}$,

$D_{\mu,s+n} = b_\mu \cdot D_{\mu+n,s+n} - a_\mu^2 \cdot D_{\mu+2n,s+n}$

by $\frac{1}{D_{\mu+n,s}}$ and $\frac{1}{D_{\mu+n,s+n}}$, respectively, and subtract them. We obtain

$\frac{D_{\mu,s}}{D_{\mu+n,s}} - \frac{D_{\mu,s+n}}{D_{\mu+n,s+n}} = a_\mu^2 \cdot \left( \frac{D_{\mu+n,s+n}}{D_{\mu+n,s}} - \frac{D_{\mu+2n,s}}{D_{\mu+n,s}} \right)$,

which proves the statement. \(\Box\)

**Lemma 2.5.** Let $f, g$ be solutions of $\tau y = 0$. If $f_j = g_j$, $\cdots$, $f_{j+n-1} = g_{j+n-1}$, $j \in \mathbb{Z}$, then

$f_{k_0} \geq g_{k_0}, \cdots, f_{k_0+n-1} \geq g_{k_0+n-1}$ for some $k_0 \geq j + n$ \(k_0 \leq j - n\) \(2\)
implies
\[ f_k > g_k \quad \forall k \geq j + n \quad (k \leq j - n). \]

**Proof.** We follow the idea introduced in [1, 2]. Let \( f \) and \( g \) be solutions of the linear system
\[
\begin{pmatrix}
y_{j+n} \\
y_{j+n+1} \\
\vdots \\
y_{h_n-1}
\end{pmatrix}
= \begin{pmatrix}
v \\
0 \\
\vdots \\
w
\end{pmatrix},
\]
where
\[
v = \begin{pmatrix}
-a_j y_j \\
\vdots \\
-a_j y_j + 1 y_{j+n-1}
\end{pmatrix},
\quad w = \begin{pmatrix}
-a_{k_0-n} y_{k_0} \\
\vdots \\
-a_{k_0-n-1} y_{k_0+n-1}
\end{pmatrix}.
\]

Then \( f_k > g_k \) for \( k \in [j + n, k_0 + n - 1] \cap \mathbb{Z} \). Now, the existence of a \( K \geq k_0 + n \) such that \( f_k < g_k \) contradicts (2).

If we suppose that \( T \geq 0 \), we can (due to Lemma 2.4, \( j \in [0, n-1] \cap \mathbb{Z} \)) introduce the limits
\[
C_{\mu,j}^+ := \lim_{i \to \infty} \frac{D_{\mu+n,s_j+i} + n}{D_{\mu,s_j+i}},
C_{\mu,j}^- := \lim_{i \to \infty} \frac{D_{\mu+n,s_j+i} - n}{D_{\mu,s_j+i}}
\]
and a positive map \( u : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R} \) defined by
\[
u \mu, k := \begin{cases}
\prod_{i=1}^{\ell} (-a_{k-in} \cdot C_+^{k-(i-1)n, k-\mu-tn}), & k \geq \mu + n, \\
1, & k \in [\mu - n + 1, \mu + n - 1], \\
\prod_{j=0}^{\ell-1} (-a_{k+in} \cdot C_-^{[k+in, k-\mu-tn]}), & k \leq \mu - n,
\end{cases}
\]
where \( \ell = \left\lfloor \frac{k-\mu}{n} \right\rfloor \), and \( \left\lfloor \cdot \right\rfloor \) denotes the floor function (greatest integer function) of a real number.

Now, let us recall the definition of a minimal solution of (1).

**Definition 2.1.** We say that a solution \( u \) of (1) is *minimal* on \([\mu + n, \infty) \cap \mathbb{Z}\) if any linearly independent solution \( v \) of (1) with \( v_\mu = u_\mu, \ldots, v_{\mu+n-1} = u_{\mu+n-1} \) satisfies \( v_k > u_k \) for \( k \geq \mu + n \). The minimal solution on \((\infty, \mu - n] \cap \mathbb{Z})\) is defined analogously.

**Lemma 2.6.** Let \( T \geq 0 \). Then \( u(\mu, k), \mu \in \mathbb{Z} \) is fixed, is the minimal positive solution of \( \tau y = 0 \) on \([\mu + n, \infty) \cap \mathbb{Z} \) and \((\infty, \mu - n] \cap \mathbb{Z} \) separately.
Proof. We introduce
\[
u_k^{[\mu, \nu]} := \begin{cases} 
1, & k \in [\mu, \mu + n - 1] \cap \mathbb{Z}, \\
(-a_{k-n})(-a_{k-2n}) \cdots (-a_{k-\ell n}) \frac{D_{k+\ell n,r_j}}{D_{k,(\ell-1)n,r_j}}, & k \in [\mu + n, s_0 - 1] \cap \mathbb{Z}, \\
(-a_{k-n})(-a_{k-2n}) \cdots (-a_{k-\ell n}) \frac{1}{D_{k,(\ell-1)n,r_j}}, & k \in [s_0, s_{n-1}] \cap \mathbb{Z}, \\
0, & k \in [s_{n-1} + 1, s_{n-1} + n] \cap \mathbb{Z},
\end{cases}
\]
where \(i = k - \mu - \ell n, \ell = \left\lfloor \frac{k-\mu}{n} \right\rfloor \).

The sequence \(u_k^{[\mu, \nu]}\) satisfies \(\tau y = 0\) on \([\mu + n, s_{n-1}] \cap \mathbb{Z}\) and it holds that
\[
\lim_{s_{n-1} \to \infty} u_k^{[\mu, \nu]} = u(\mu, \nu), \quad k \geq \mu,
\]
so we can see that \(\tau u = 0\) on \([\mu + n, \infty) \cap \mathbb{Z}\).

Let \(v\) be a positive solution of \(\tau y = 0\) on \([\mu + n, \infty) \cap \mathbb{Z}\) such that \(v_\mu = 1, \ldots, v_{\mu+n-1} = 1\), which is linearly independent on \(u(\mu, \cdot)\).

Because \(v_{s_{n-1}+1} > 0, \ldots, v_{s_{n-1}+n} > 0\), we have \(v_k > u_k^{[\mu, s_{n-1}]}\) on \([s_{n-1} + 1, s_{n-1} + n] \cap \mathbb{Z}\). Thus, by Lemma 2.5, \(v_k > u_k^{[\mu, s_{n-1}]}\) for all \(k \geq \mu + n\). Hence \(v_k \geq u(\mu, k)\) for all \(k \geq \mu\). Assume, by contradiction, that there exists \(k_0 \geq \mu + n\) such that \(v_{k_0} = u(\mu, k_0)\). Then, again by Lemma 2.5, \(v_k = u(\mu, k)\) for all \(k \geq \mu\), which is a contradiction.

One can show analogously that \(u\) is the minimal positive solution of \(\tau y = 0\) on \((-\infty, \mu - n] \cap \mathbb{Z}\) using
\[
u_k^{[\mu, \nu]} := \begin{cases} 
1, & k \in [\mu - n + 1, \mu] \cap \mathbb{Z}, \\
(-a_k)(-a_{k+n}) \cdots (-a_{k+(\ell-1)n}) \frac{D_{r_{j-1}, k-\ell n}}{D_{r_{j-1}, k+(\ell-1)n}}, & k \in [r_0 + 1, \mu - n] \cap \mathbb{Z}, \\
(-a_k)(-a_{k+n}) \cdots (-a_{k+(\ell-1)n}) \frac{1}{D_{r_{j-1}, k+(\ell-1)n}}, & k \in [r_{n-1}, r_0] \cap \mathbb{Z}, \\
0, & k \in [r_{n-1} - 1, r_{n-1} - n] \cap \mathbb{Z},
\end{cases}
\]
where \(j = \mu - k - \ell n, \ell = \left\lfloor \frac{k-\mu}{n} \right\rfloor \).

3. Positive solutions of \(\tau y = 0\)

If \(T \geq 0\), as a consequence of the previous section, the minimal positive solutions of the equation \(\tau y = 0\) on \([n, \infty) \cap \mathbb{Z}\) and \((-\infty, n] \cap \mathbb{Z}\) are
\[
u_k^n := \begin{cases} 
u(0, k), & k \in \mathbb{N}_0, \\
u(k - i, i)^{-1}, & k \equiv i \pmod{n}, \quad -k \in \mathbb{N}, \quad i \in [0, n - 1] \cap \mathbb{Z},
\end{cases}
\]
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\[ u_k := \begin{cases} 
  u(0,k), & -k \in \mathbb{N}_0, \\
  u(k+i,-i)^{-1}, & k \equiv i \pmod{n}, \ k \in \mathbb{N}, \ i \in [0,n-1] \cap \mathbb{Z}, 
\end{cases} \]

respectively.

**Lemma 3.1.** Let \( T \geq 0 \). Then \( u_k^+ \) and \( u_k^- \) are positive solutions of \( \tau y = 0 \) on \( \mathbb{Z} \).

**Proof.** We show that \( u_k^+ \) is a positive solution of \( \tau y = 0 \) on \( \mathbb{Z} \), the statement for \( u_k^- \) can be proved similarly.

The sequence \( u^+ \) is a positive solution of \( \tau y = 0 \) on \( [n, \infty) \). It follows from the definition of \( u(\mu,k) \) that for

\[ \forall k, \ell \in \mathbb{Z}, \ \forall m \in [k, \ell] \cap \mathbb{Z}, \ m \equiv k \pmod{n} \]

it holds that

\[ u(k, \ell) = u(k,m) \cdot u(m, \ell). \]

Let \( m \in (-\infty, n-1] \cap \mathbb{Z} \) be arbitrary and \( M \in -\mathbb{N} \) such that \( M \leq m - n \) be fixed. Then for some \( i \in [0, n-1] \cap \mathbb{Z} \) we have \( M \equiv m - i \pmod{n} \) and

\[ u(M, i) = u(M, m-i) \cdot u(m-i, i). \]

Hence, by the definition of \( u^+ \), we obtain

\[ u^+_m = u(m-i, i)^{-1} = \frac{u(M, m-i)}{u(M, i)}, \]

which implies that \( u^+ \) is a solution of \( \tau y = 0 \) on \( [M + i, i] \). \( \square \)

**Theorem 3.1.** \( T \geq 0 \) if and only if there exists a positive solution of \( \tau y = 0 \).

**Proof.** The necessity follows from Lemma 3.1. To prove sufficiency we show at first positivity of all determinants \( D_{\mu, \xi} \), \( \mu \leq \xi \leq \nu \), \( \xi \equiv \mu \pmod{n} \). Assume that there exists a positive solution \( u \) of \( \tau y = 0 \). Then \( u \) solves the system

\[
\begin{pmatrix}
  u_\mu \\
  u_{\mu+1} \\
  \vdots \\
  u_\nu
\end{pmatrix}
= t_{\mu, \nu}
\begin{pmatrix}
  v \\
  0 \\
  \vdots \\
  0 \\
  w
\end{pmatrix},
\]

where

\[
 v = \begin{pmatrix}
  -a_{\mu-n}u_{\mu-n} \\
  \vdots \\
  -a_{\mu-1}u_{\mu-1}
\end{pmatrix}, \quad w = \begin{pmatrix}
  -a_{\nu-n+1}u_{\nu+1} \\
  \vdots \\
  -a_{\nu}u_{\nu+n}
\end{pmatrix}.
\]
By Cramer's rule we obtain

\[
u_{\mu}d_{\mu,\nu} = \left[ -a_{\mu-n}u_{\mu-n}D_{\mu+n,s_0} \\
+ (-a_{\sigma}u_{\sigma+n})(-a_{\mu+n})\cdots(-a_{s_0-n}) \right] \left( \prod_{i=1}^{n-1} D_{\mu+i,s_i} \right),
\]

where \( \sigma \in \{ \nu - n + 1, \ldots, \nu \} \), \( \sigma \equiv \mu \pmod{n} \). Hence

\[
D_{\mu,s_0}u_{\mu} = -a_{\mu-n}u_{\mu-n}D_{\mu+n,s_0} + (-a_{\mu})(-a_{\mu+n})\cdots(-a_{s_0-n})(-a_{\sigma})u_{\sigma+n}.
\]

If \( s_0 = \mu + n \) then \( D_{\mu+n,s_0} = D_{\mu+n+n} = b_{\mu+n} > 0 \) which implies that \( D_{\mu+n+n} > 0 \). Hence all determinants \( D_{\mu,\xi} \) are positive.

Similarly we can show positivity of the determinants \( D_{\mu+i,\xi,} \), \( \mu + i \leq \xi, \xi, \equiv \mu + i \pmod{n} \), \( i = 1, \ldots, n - 1 \), which gives positivity of the determinants \( d_{\mu,\nu}, \mu \leq \nu \) and the statement follows from Lemma 2.3.

In an analogous way as before we can generalize a number of statements of [1]. For example, the following result, which characterizes the minimal solutions of (1), can be proved in the same way as in [1] (see also [3]).

**Theorem 3.2.** If \( T \geq 0 \), then for all \( \mu \in \mathbb{Z} \)

\[
\sum_{i=\mu}^{\infty} (-a_{i}u_{i}^{\pm}u_{i+n}^{\pm})^{-1} = \infty.
\]

**References**


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SOME NUMERICAL INVESTIGATIONS OF DIFFERENTIAL EQUATIONS WITH SEVERAL PROPORTIONAL DELAYS

J. JáNSKÝ

Abstract. This paper describes the application and asymptotics of the trapezoidal rule for the pantograph differential equation with several proportional delays. Some comparisons with known results are given as well.

INTRODUCTION

We shall study some numerical properties of the delay differential equation

\[ y'(t) = ay(t) + \sum_{i=1}^{k} b_i y(\lambda_i t), \quad t \geq 0, \]

where \( a < 0, b_i \neq 0 \) and \( 0 < \lambda_i < 1 \) are real scalars, \( i \in \{1, 2, \ldots, k\} \). This equation is the pantograph equation with several delays. The “classical” pantograph equation was derived in 1970 in the frame of solving of problem on the railways. Later, many other applications in various areas led to this equation and its modifications. Therefore this equation has become the subject of many qualitative and numerical investigations. The pantograph equation and its modifications were studied in the continuous case \([2, 3, 5, 6, 9]\) as well as in the discrete case \([1, 4, 7, 12, 10]\).

We focus on difference equations arising from (1) by use of the numerical discretization. The Euler method \([8, 11]\) and the trapezoidal rule \([1, 4]\) represent the standard ways of such discretization.

In this paper we wish to describe some asymptotic properties of the trapezoidal rule discretization of (1).

1. Trapezoidal rule for the equation (1)

In this section, we outline a discretization of equation (1) by use of the trapezoidal rule. Note that an analogous problem was discussed in \([1]\). After integration of (1)
we obtain
\[ y(t) = y(0) + a \int_0^t y(\tau) \, d\tau + \sum_{i=1}^k b_i \int_0^t y(\lambda_i \tau) \, d\tau. \]

Now we introduce the substitution \( u_i = \lambda_i \tau \) in the second integral and get
\[ y(t) = y(0) + a \int_0^t y(\tau) \, d\tau + \sum_{i=1}^k b_i \frac{\lambda_i}{\lambda_i n} \int_0^{\lambda_i n} y(u_i) \, du_i. \]

Considering the grid of points \((nh), n = 0,1, \ldots\) with stepsize \( h > 0 \) we can write
\[ y((n+1)h) = y(nh) + a \int_{nh}^{(n+1)h} y(\tau) \, d\tau + \sum_{i=1}^k b_i \frac{\lambda_i}{\lambda_i n} \int_{\lambda_i nh}^{\lambda_i (n+1)h} y(u_i) \, du_i. \]

Now we approximate both integrals using the trapezoidal rule. The approximation of the first integral is simple:
\[ \int_{nh}^{(n+1)h} y(\tau) \, d\tau \approx \frac{1}{2} h (y_n + y_{n+1}), \]
where \( y_n \approx y(nh) \). From the second integral, we obtain
\[ \frac{b_i}{\lambda_i} \int_{\lambda_i nh}^{\lambda_i (n+1)h} y(u_i) \, du_i \approx \frac{h b_i}{2} \left( y(\lambda_i nh) + y(\lambda_i (n+1)h) \right). \]

Since the points \( \lambda_i nh \) are generally not in the given grid, we have to approximate them using a linear interpolation. Hence
\[
\begin{align*}
y_{\lambda_i n} &\approx \left( 1 - (\lambda_i n - \lfloor \lambda_i n \rfloor) \right) y_{\lfloor \lambda_i n \rfloor} + (\lambda_i n - \lfloor \lambda_i n \rfloor) y_{\lfloor \lambda_i n \rfloor + 1}, \\
y_{\lambda_i (n+1)} &\approx \left( 1 - (\lambda_i (n+1) - \lfloor \lambda_i n \rfloor) \right) y_{\lfloor \lambda_i n \rfloor} + (\lambda_i (n+1) - \lfloor \lambda_i n \rfloor) y_{\lfloor \lambda_i n \rfloor + 1}.
\end{align*}
\]

Overall we have
\[ y_{n+1} = y_n + \frac{1}{2} h (y_n + y_{n+1}) + h \sum_{i=1}^k b_i \left( \beta_{n,i} y_{\lfloor \lambda_i n \rfloor} + \alpha_{n,i} y_{\lfloor \lambda_i n \rfloor + 1} \right), \tag{2} \]
where
\[ \alpha_{n,i} := \lambda_i n - \lfloor \lambda_i n \rfloor + \frac{\lambda_i}{\lambda_i n} > 0, \quad \beta_{n,i} := 1 - \alpha_{n,i}. \]

From formula (2) we express \( y_{n+1} \) and get the final form of the discretized equation (1)
\[ y_{n+1} = Ry_n + Sh \sum_{i=1}^k b_i \left( \beta_{n,i} y_{\lfloor \lambda_i n \rfloor} + \alpha_{n,i} y_{\lfloor \lambda_i n \rfloor + 1} \right), \tag{3} \]
where
\[ R := \frac{2+ah}{2-ah}, \quad S := \frac{2}{2-ah}. \]
2. Some auxiliary results

In this section we present the inequality which is useful in our further calculations. This inequality has the form:

\[ |S|h \sum_{i=1}^{k} |b_i| (|\beta_{n,i}|p_{\lambda,n} + |\alpha_{n,i}|p_{\lambda,n+1}) \leq (1 - |R|)\rho_n, \quad n = 0, 1, \ldots \]  \hspace{1cm} (4)

Further, we have

\[ \eta_i := \sup_{n \in \mathbb{Z}^+} (|\beta_{n,i}| + |\alpha_{n,i}|) < \infty \quad \text{and} \quad |R| < 1, \]  \hspace{1cm} (5)

because of the assumptions \( a < 0, h > 0 \).

**Lemma 2.1.** Let (5) holds. Then the sequence

\[ \rho_n := \left\{ \begin{array}{ll} (n - \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}} & \text{for } \tilde{\gamma} \geq 1, \\ (n + \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}} & \text{for } 0 < \tilde{\gamma} < 1, \end{array} \right. \]  \hspace{1cm} (6)

where

\[ \lambda := \left\{ \begin{array}{ll} \max(\lambda_1, \lambda_2, \ldots, \lambda_k) & \text{for } \tilde{\gamma} \geq 1, \\ \min(\lambda_1, \lambda_2, \ldots, \lambda_k) & \text{for } 0 < \tilde{\gamma} < 1 \end{array} \right. \]  \hspace{1cm} (7)

and

\[ \tilde{\gamma} := \frac{h|S| \sum_{i=1}^{k} |b_i| \eta_i}{1 - |R|} \]  \hspace{1cm} (8)

is a solution of inequality (4).

**Proof.** We only deal with the case \( \tilde{\gamma} < 1 \) because the case \( \tilde{\gamma} \geq 1 \) is analogical. If \( \tilde{\gamma} < 1 \), then \( (\rho_n) \) is a decreasing sequence. Hence we can write

\[ |S|h \sum_{i=1}^{k} |b_i| (|\beta_{n,i}|p_{\lambda,n} + |\alpha_{n,i}|p_{\lambda,n+1}) \leq |S|h \sum_{i=1}^{k} \eta_i |b_i| p_{\lambda,n}. \]

Further

\[ |S|h \sum_{i=1}^{k} \eta_i |b_i| p_{\lambda,n} = |S|h \sum_{i=1}^{k} \eta_i |b_i| (|\lambda,n| + \frac{1}{\lambda})^{-\log_\lambda \tilde{\gamma}} \leq \]

\[ \leq |S|h \sum_{i=1}^{k} \eta_i |b_i| (\lambda,n - 1 + \frac{1}{\lambda})^{-\log_\lambda \tilde{\gamma}} = |S|h \sum_{i=1}^{k} \eta_i |b_i| (\lambda,n + \frac{1}{\lambda})^{-\log_\lambda \tilde{\gamma}} \]

\[ \leq |S|h \sum_{i=1}^{k} \eta_i |b_i| (\lambda,n + \frac{1}{\lambda})^{-\log_\lambda \tilde{\gamma}} = |S|h \sum_{i=1}^{k} \eta_i |b_i| \lambda^{-\log_\lambda \tilde{\gamma}} \rho_n = (1 - |R|)\rho_n. \] \hspace{1cm} \( \square \)
3. Main result

The main theorem of this paper is the following.

**Theorem 3.1.** Let \((y_n)\) be a solution of (3), let \(a < 0, b_i \neq 0\) for all \(i \in \{1, 2, \ldots, k\}\) and let \((\rho_n), \lambda, \tilde{\gamma}\) be given by (6)–(8). Then

\[
|y_n| \leq Kn^{-\log_{1/\lambda} \tilde{\gamma}} \quad \text{for} \quad n = \sigma_0, \sigma_0 + 1, \sigma_0 + 2, \ldots,
\]

where \(\sigma_0 \in \mathbb{Z}^+\) satisfies \(\sigma_0 \geq \left\lceil \frac{2}{(1 - \lambda)\tilde{\gamma}} \right\rceil 2 \log_{1/\lambda} \tilde{\gamma}\) and \(K := B_0 \exp(\frac{1}{1 - \lambda})\) with

\[
B_0 := \sup \left( \frac{\rho_n}{\rho_m}, n \in [\lambda(\sigma_0 - 1), \sigma_0] \cap \mathbb{Z}^+ \right)
\]

and

\[
L := \frac{2 \log_{1/\lambda} \tilde{\gamma}}{\tilde{\gamma}(1 - |R|)(\sigma_0 - \frac{1 + \tilde{\gamma}}{1 - \lambda})}.
\]

**Proof.** We use the substitution \(z_n = \frac{\rho_n}{\rho_m}\) in (3), where \(\rho_n\) is given by (6). Then

\[
\varrho_{n+1}z_{n+1} = R\rho_nz_n + Sh \sum_{i=1}^{k} b_i \left( \beta_{n,i} \rho_{\lfloor \lambda n \rfloor} z_{\lfloor \lambda n \rfloor} + \alpha_{n,i} \rho_{\lfloor \lambda n \rfloor + 1} z_{\lfloor \lambda n \rfloor + 1} \right).
\]

Now we choose \(\sigma_0 \geq \left\lceil \frac{2}{(1 - \lambda)\tilde{\gamma}} \right\rceil 2 \log_{1/\lambda} \tilde{\gamma}\), \(\sigma_0 \in \mathbb{Z}^+\) and define points

\[
\sigma_{m+1} := \left\lceil \frac{\sigma_m}{1 - 1/\lambda} \right\rceil, \quad m = 0, 1, \ldots
\]

After some calculations, we obtain

\[
\lambda^{-m}(\sigma_0 - \frac{1 + \tilde{\gamma}}{1 - \lambda}) \leq \sigma_m \leq \lambda^{-1}\sigma_{m-1}, \quad m = 1, 2, \ldots
\]

Next we introduce intervals \(I_0 := [\lambda(\sigma_0 - 1), \sigma_0] \cap \mathbb{Z}^+, I_{m+1} := [\sigma_m, \sigma_{m+1}] \cap \mathbb{Z}^+\) and denote \(B_m := \sup(|z_s|, s \in \bigcup_{j=0}^{m} I_j), m = 0, 1, 2, \ldots\).

Now we choose \(n^* \in I_{m+1}, n^* > \sigma_m\) arbitrarily and we distinguish two cases with respect to \(R\).

(i) First, we deal with the case \(R = 0\). In this case

\[
z_{n^*} = \frac{Sh}{\rho_{n^*}} \sum_{i=1}^{k} b_i \left( \beta_{n^*,i} \rho_{\lfloor \lambda n^* \rfloor} z_{\lfloor \lambda n^* \rfloor} + \alpha_{n^*,i} \rho_{\lfloor \lambda n^* \rfloor + 1} z_{\lfloor \lambda n^* \rfloor + 1} \right),
\]

hence

\[
|z_{n^*}| \leq B_m \frac{|Sh|}{\rho_{n^*}} \sum_{i=1}^{k} |b_i| \left( |\beta_{n^*,i} \rho_{\lfloor \lambda n^* \rfloor}| + |\alpha_{n^*,i} \rho_{\lfloor \lambda n^* \rfloor + 1}| \right).
\]

Using (4), we arrive at

\[
|z_{n^*}| \leq \frac{\rho_{n^*,m}}{\rho_{n^*}} B_m.
\]

Assuming \(\tilde{\gamma} \geq 1\), \((\rho_n)\) is the nondecreasing sequence and we obtain \(|z_{n^*}| \leq B_m\). Assuming \(0 < \tilde{\gamma} < 1\) we derive with respect to (6), (12) and the binomial formula the relation

\[
\frac{\rho_{n^*,m}}{\rho_{n^*}} = \left( \frac{n^* + 1 - \lambda}{n^*} \right) - \log_{1/\lambda} \tilde{\gamma} \leq \frac{1}{(1 + \frac{1}{\sigma_m} - \log_{1/\lambda} \tilde{\gamma})} \leq \frac{1}{1 + \frac{1 - \log_{1/\lambda} \tilde{\gamma}}{\sigma_m}} \leq 1 + \frac{2 \log_{1/\lambda} \tilde{\gamma}}{\sigma_m}.
\]
This inequality implies the following relation
\[ |z_{n^*}| \leq B_m \left( 1 + \frac{2 \log \gamma}{\sigma_0 - \frac{1}{\lambda}} \right). \tag{13} \]

(ii) Let \( R \neq 0 \). We can multiply the equation (11) by \( \frac{1}{R}|z_{n^*}| \). We get
\[ \Delta \left( \frac{\partial z_{n^*}}{\partial \sigma} \right) = \sum_{i=1}^{k} b_i \left( \beta_{i,\sigma}[\lambda_i \sigma_{n^*}] z_{\lambda_i [n^*]} + \alpha_{i,\sigma}[\lambda_i \sigma_{n^*}] z_{\lambda_i [n^*] + 1} \right). \]

If we sum this relation from \( \sigma_m \) to \( n^* - 1 \) than we obtain
\[ \frac{\partial z_{n^*}}{\partial \sigma} = \sum_{p = \sigma_m}^{n^* - 1} \frac{R_{n^* - \sigma_m}}{\delta_{n^*}} \sum_{i=1}^{k} b_i \left( \beta_{p,i}[\lambda_i \sigma_{n^*}] z_{\lambda_i [p]} + \alpha_{p,i}[\lambda_i \sigma_{n^*}] z_{\lambda_i [p] + 1} \right). \]

Thus
\[ |z_{n^*}| \leq B_m \left( \frac{\sigma_m}{\delta_{n^*}} |R|^{n^* - \sigma_m} + \frac{|R|^{n^*}}{\delta_{n^*}} \sum_{p = \sigma_m}^{n^* - 1} \sum_{i=1}^{k} b_i \left( |\beta_{p,i}[\lambda_i \sigma_{n^*}]| + |\alpha_{p,i}[\lambda_i \sigma_{n^*}]| \right) \right). \]

Using (4), we get
\[ |z_{n^*}| \leq B_m \left( \frac{\sigma_m}{\delta_{n^*}} |R|^{n^* - \sigma_m} + \frac{|R|^{n^*}}{\delta_{n^*}} \sum_{p = \sigma_m}^{n^* - 1} \frac{1 - |R|}{|R|^{p+1}} \rho_p \right). \]

Now using the relation
\[ \frac{1 - |R|}{|R|^{p+1}} = \Delta \left( \frac{1}{|R|} \right)^p \tag{14} \]
and summing by parts we get
\[ |z_{n^*}| \leq B_m \left( \frac{\sigma_m}{\delta_{n^*}} |R|^{n^* - \sigma_m} + \frac{|R|^{n^*}}{\delta_{n^*}} \sum_{p = \sigma_m}^{n^* - 1} \Delta \left( \frac{1}{|R|} \right)^p \rho_p \right) \]
\[ = B_m \left( \frac{\sigma_m}{\delta_{n^*}} |R|^{n^* - \sigma_m} + 1 - \frac{\sigma_m}{\delta_{n^*}} |R|^{n^* - \sigma_m} - \frac{|R|^{n^*}}{\delta_{n^*}} \sum_{p = \sigma_m}^{n^* - 1} \frac{1}{|R|^{p+1}} \Delta \rho_p \right) \]
\[ = B_m \left( 1 - \frac{|R|^{n^*}}{\delta_{n^*}} \sum_{p = \sigma_m}^{n^* - 1} \frac{\Delta \rho_p}{1 - |R|} \Delta \left( \frac{1}{1 - |R|} \right)^p \right). \]
If $\gamma \geq 1$ then $\rho_p$ is nondecreasing, therefore $\Delta \rho_p \geq 0$ and $|z_n^*| \leq B_m$. In the case $0 < \gamma < 1$, same simple calculations are necessary to derive that $\Delta \rho_p$ is negative and nondecreasing. Hence we can write

$$|z_n^*| \leq B_m \left( 1 - \frac{|R||\sigma_n^*|}{\sigma_m} \sum_{p=\sigma_m}^{n^*-1} \Delta \left( \frac{1}{|R|} \right) \right) = B_m \left( 1 - \frac{|R||\sigma_n^*|}{\sigma_m} \Delta \left( \frac{1}{|R|} \right) \right)$$

$$\leq B_m \left( 1 + \frac{1}{|R|} \Delta \rho_{\sigma_m} \right) \leq B_m \left( 1 + \frac{1}{|R|} \Delta \rho_{\sigma_{m+1}} \right).$$

Substituting the corresponding form of $\rho_n$ (see (6)) and using the binomial formula, we can derive

$$-\Delta \rho_{\sigma_m} = \rho_{\sigma_m} - \rho_{\sigma_{m+1}} = \left( \sigma_m + \frac{1}{1-\lambda} \right)^{\log_\gamma \delta} - \left( \sigma_m + 1 + \frac{1}{1-\lambda} \right)^{\log_\gamma \delta}$$

$$= \left( \sigma_m + \frac{1}{1-\lambda} \right)^{\log_\gamma \delta} - \left( 1 + \frac{1}{1-\lambda} \right)^{\log_\gamma \delta}$$

$$\leq \left( \sigma_m + \frac{1}{1-\lambda} \right)^{\log_\gamma \delta} \left( 1 + \frac{1}{\sigma_m} \right)^{\log_\gamma \delta} \leq \left( \sigma_m + \frac{1}{1-\lambda} \right)^{\log_\gamma \delta}.$$ 

and analogically

$$\rho_{\sigma_{m+1}} = \left( \sigma_{m+1} + \frac{1}{1-\lambda} \right)^{\log_\gamma \delta} \geq \left( \frac{1}{\lambda} \sigma_{m+1} + \frac{1}{1-\lambda} \right)^{\log_\gamma \delta}$$

$$\geq \left( \frac{1}{\lambda} \sigma_{m+1} + \frac{1}{1-\lambda} \right)^{\log_\gamma \delta} \geq \left( \sigma_{m+1} + \frac{1}{1-\lambda} \right)^{\log_\gamma \delta}.$$ 

Considering (12) we arrive at

$$-\Delta \rho_{\sigma_{m+1}} \leq \log_\gamma \delta \frac{1}{\sigma_m} \leq \frac{1}{\sigma_m} \leq \log_\gamma \delta \left( \sigma_m + \frac{1}{1-\lambda} \right)^{\log_\gamma \delta}.$$ 

Hence

$$|z_n^*| \leq B_m \left( 1 + \frac{1}{|R|} \lambda^m \right).$$

(15)

Using the definition of $L$, summarizing cases (i)–(ii) and using estimates (13) and (15) we get

$$|z_n^*| \leq B_m \left( 1 + L \lambda^m \right) \quad \text{as} \quad m \to \infty$$

for arbitrary $n^* \in I_{m+1}$, $n^* > \sigma_m$. Thus

$$B_{m+1} \leq B_m \left( 1 + L \lambda^m \right) \quad \text{as} \quad m \to \infty.$$ 

Now we can estimate $B_m$ in this way:

$$B_{m+1} \leq B_m \left( 1 + L \lambda^m \right) \leq B_0 \prod_{j=0}^m \left( 1 + L \lambda^j \right) \leq B_0 \prod_{j=0}^\infty \left( 1 + L \lambda^j \right) \leq B_0 \exp \left( L \frac{1}{1-\lambda} \right).$$

Thus

$$B_m \leq B_0 \exp \left( L \frac{1}{1-\lambda} \right) \quad \text{as} \quad m \to \infty.$$ 

The estimate (9) is proved. \qed

**Remark 3.2.** The constant $\sigma_0$ has to be proposed with respect to a concrete equation. If we choose $\sigma_0 \geq \max \left\{ \frac{2}{(1-\lambda)\lambda}, 2 \log_\gamma \delta \right\}$ too small, then the constant $L$ can be too large and the estimate (9) becomes worse. If we choose $\sigma_0$ too large, then it will be necessary to calculate the constant $B_0$ in (10) in too large interval.
Example 3.3. In this example we show the application of Theorem 3.1. Let us consider the following initial value problem
\[ y'(t) = -y(t) - 0.25y(\frac{t}{4}) - 0.2y(\frac{t}{4}), \quad t \geq 0, \quad y(0) = 1. \] (16)

After a discretization (3) with the stepsize \( h = 0.05 \) we obtain
\[ y_0 = 1, \]
\[ y_{n+1} = \frac{34}{41}y_n - \frac{1}{82}(\beta_{n,1}y_{[n/4]} + \alpha_{n,1}y_{[n/4]+1}) - \frac{2}{205}(\beta_{n,2}y_{[n/3]} + \alpha_{n,2}y_{[n/3]+1}), \]
where
\[ \alpha_{n,1} := \frac{n}{4} - \lfloor \frac{n}{4} \rfloor + 1 > 0, \quad \beta_{n,1} := 1 - \alpha_{n,1}, \quad \alpha_{n,2} := \frac{n}{3} - \lfloor \frac{n}{3} \rfloor + 1 > 0, \quad \beta_{n,2} := 1 - \alpha_{n,2}. \]

Now if we set \( \sigma_0 = 488 \), then using Theorem 3.1 we obtain the estimate
\[ |y_n| \leq 2.138n^{-0.576}, \quad \text{for} \quad n = 488, 489, \ldots \] (17)

The Fig. 1 displays the real numerical solution of the problem (16) and its estimate given by (17).

4. SOME COMPARISONS

In this section we compare our results with results given in [6] and [1]. For the sake of simplicity we restrict our comparisons to the case \( k = 1 \) and denote \( b := b_1 \). Under these conditions the equation (1) is reduced to the scalar pantograph equation
\[ y'(t) = ay(t) + by(\lambda t), \quad t \geq 0, \] (18)
where \( \lambda = \lambda_1 \) and assume \( a < 0, b \neq 0 \).

First we present the estimate of the exact equation (18). It was proved in [6] that the asymptotic estimate
\[ y(t) = O(t^{-\log_{|\lambda^3|}}) \quad \text{as} \quad t \to \infty \]
holds for any solution of (18). Furthermore it is known that the constant \(- \log_{\lambda} \frac{b}{a}\) is the best one and cannot be improved.

If we denote \(\alpha_n := \alpha_{n,1}\) and \(\beta_n := \beta_{n,1}\) then the difference equation (3) becomes
\[
y_{n+1} = Ry_n + Shb \left( \beta_n y_{[\lambda n]} + \alpha_n y_{[\lambda n]+1} \right).
\]
Now we denote \(\eta := \eta_1\). It was shown in [1] that any solution \(y_n\) of (19) fulfilling
\[
|R| + h\eta|S||b| \leq 1,
\]
\(a < 0\)
is bounded, and the asymptotic estimate
\[
y_n = O(n^{-\log_{\lambda} \gamma}), \quad \gamma := |R| + h\eta|S||b|, \quad \text{as } n \to \infty
\]
holds for any solution of (19).

It is interesting to compare the asymptotic estimate (20) with (9). It can be shown that if \(\gamma < 1\) then
\[
\tilde{\gamma} = \frac{h\eta|Sb|}{1 - |R|} < |R| + h\eta|Sb| = \gamma.
\]
Moreover, \(\tilde{\gamma}\) can be expressed as
\[
\tilde{\gamma} = \frac{1}{2} \eta \quad \text{for} \quad h|a| \leq 2, \quad \tilde{\gamma} = \frac{h|b|}{2} \quad \text{for} \quad h|a| > 2.
\]
It is known (see [1, Theorem 6]) that if \(\lambda = \frac{1}{2}\) where \(l \in \{2, 3, \ldots\}\) then \(\eta = 1\).

Hence we get the value \(\frac{1}{2}\) known from the asymptotic estimate of the solution of the exact equation (18) provided \(h|a| \leq 2\).

\section*{References}

INVESTIGATING OF BOUNDARY VALUE PROBLEMS FOR
ORDINARY DIFFERENTIAL EQUATIONS IN A CRITICAL
CASE

IHOR KOROL

Abstract. The numerical-analytic method for investigating and approximate con-
structing of the solutions of boundary value problems for nonlinear differential sys-
tems in a critical case is suggested.

1. Introduction

The theory of boundary value problems (BVP) is an important branch of the
genral theory of differential equations due to their strong connection with the
practical applications. Various types of methods (numerical, analytic, numerical-
analytic, functional etc) [1–7] for investigating the problems of existing and ap-
proximate constructing of the solutions are studied.

The modification of the numerical-analytic method, which is developed in this
paper allow us to study BVP

\[ \frac{dx}{dt} = A(t)x + f(t, x), \quad B_1x(a) + B_2x(b) + \int_a^b [dB(t)]x(t) = d \]

in a critical [2] case – when there exist a nontrivial solutions of the correspondence
linear homogeneous problem. We can use this modification when the boundary
condition is "degenerate" [7]. Also we can note that due to using this modifi-
cation the restriction for Lipschitz matrix concerns not the whole right part of the
differential equation, but only the nonlinearity \( f(t, x) \), and this is less difficult to
deal with.

2. Linear BVP

In the beginning let us consider a linear inhomogeneous system of differential
equations

\[ \frac{dx}{dt} = A(t)x + b(t), \]

Key words and phrases. ordinary differential equations, boundary value problems, numerical-
analytic method, critical case, successive approximations.
with the additional linear condition, which can be represent by means of Riemann-
Stieltjes integral

\[ B_1x(a) + B_2x(b) + \int_a^b [dB(t)]x(t) = d. \quad (2) \]

Let us assume that matrix \( A(t) : [a, b] \to \mathbb{R}^{n \times n} \) and function \( h(t) : [a, b] \to \mathbb{R}^n \)
are continuous, \( B_1, B_2 \in \mathbb{R}^{n \times n} \) are the constant matrices, matrix \( B(t) : [a, b] \to \mathbb{R}^{n \times n} \) is of bounded variation, \( d \in \mathbb{R}^n \).

A continuously differentiable \( n \)-dimensional function \( x \in C^1([a, b], \mathbb{R}^n), t \in [a, b] \)
is said to be a solution of the problem (1), (2) if it satisfies the equation (1) and verifies also the boundary condition (2).

It is known that the solution \( x(t, x_0) \) of the differential system (1) with the
initial value \( x(a) = x_0 \) is of the form

\[ x(t, x_0) = \Omega^t_a x_0 + \int_a^t \Omega^t_s h(s) ds, \quad (3) \]

where \( \Omega^t_a, \Omega^a_a = I_n \) is a matriciant of the corresponding to (1) homogeneous system
of differential equations

\[ \frac{dx}{dt} = A(t)x, \quad (4) \]

\( I_n \) is identity \((n \times n)\)-matrix.

While substituting (3) into the boundary condition (2) one can see that \( x(t, x_0) \)
satisfies the boundary condition (2) if and only if \( x_0 \) is a solution of the algebraic system

\[ Gx_0 = d - \int_a^b W(s)h(s) ds, \quad (5) \]

where

\[ W(s) = B_2\Omega^b_s + \int_s^b [dB(t)]\Omega^t_s, \quad G = B_1 + W(a) = B_1 + B_2\Omega^b_a + \int_a^b [dB(t)]\Omega^t_a. \quad (6) \]

Obviously that in a noncritical case [2] – when a correspondent to (1), (2)
homogeneous BVP

\[ \frac{dx}{dt} = A(t)x, \quad B_1x(a) + B_2x(b) + \int_a^b [dB(t)]x(t) = 0 \quad (7) \]
does not have the nontrivial solutions, the algebraic system (5) has a unique solution

\[ x_0 = G^{-1}\left( d - \int_a^b W(s)h(s) ds \right), \]
which is the initial value of a unique solution of the BVP (1), (2)

\[ x(t) = \Omega^1_t G^{-1} d - \Omega^1_t G^{-1} \int_a^b W(s) h(s) ds + \int_a^t \Omega^1_s h(s) ds. \]

Let us consider the problem of existence of the solutions of the BVP (1), (2) in a critical case [2] i.e. when

A) the corresponding homogeneous BVP (7) has \( k \) nontrivial linearly independent solutions, \( k = n - \text{rank}(G), 1 \leq k \leq n \).

**Lemma 2.1.** Assume that the linear homogeneous BVP (2), (4) has \( k \) linearly independent solutions. Then for an arbitrary function \( h(t) \) there exist a function \( H(t) \) such that inhomogeneous differential system

\[ x' = A(t)x + h(t) + H(t), \quad (8) \]

possesses a \( k \)-parametric family of solutions, which satisfies the boundary condition (2).

**Proof.** From (5) we have [2, 4] that the BVP (8), (2) has a solution if and only if the condition

\[ P_G \left( d - \int_a^b W(s) \left( h(s) + H(s) \right) ds \right) = 0 \quad (9) \]

is fulfilled, where \( P_G \) is an orthoprojector from the space \( \mathbb{R}^n \) to the null space \( \ker(G) \) and \( P_G^* \) is an orthoprojector from the space \( \mathbb{R}^n \) to the null space \( \ker(G^*) \):

\[ P_G : \mathbb{R}^n \to \ker(G), \quad \ker(G) = \{ y : y \in \mathbb{R}^n, Gy = 0 \}, \]
\[ P_G^* : \mathbb{R}^n \to \ker(G^*), \quad \ker(G^*) = \{ z : z \in \mathbb{R}^n, zG^* = 0 \}, \]
\[ \text{rank} P_G = \text{rank} P_G^k = \text{rank} P_G^* = \text{rank} P_{G^k} = k. \]

We will denote by \( P_{G^k} \), the \((n \times k)\)-matrix, which columns constituting a basis of the kernel \( \ker(G) \) and they are the linearly independent columns of \( P_G \). Respectively by \( P_{G^*} \) we denote \((k \times n)\)-matrix, which rows constituting a basis of the \( \ker(G^*) \) and are the linearly independent rows of matrix \( P_{G^*} \).

Let us denote

\[ H(t) = W^*(t)(P_{G^k})^* R_1^{-1} P_{G^k} \left( d - \int_a^b W(s) h(s) ds \right), \quad (10) \]

where

\[ R_1 = P_{G^k} R_2 (P_{G^k})^*, \]
\[ R_2 = \int_a^b W(s) W^*(s) ds = B_2 \int_a^b \Omega^1_s W^*(s) ds + \int_a^t [dB(t)] \int_a^t \Omega^1_s W^*(s) ds. \quad (11) \]

Substituting (10) into (9) we obtain
where \( G = \text{rank} G \) will obtain the zero vector. It means that system (13) is solvable and (because \( x, \xi \) the general 

Substituting \( x(t, x_0) \) of the form (12) into the boundary condition (2) we can see that \( x(t, x_0) \) satisfies the boundary condition (2) if and only if the initial value \( x_0 \) is a solution of the algebraic system

\[
Gx_0 = d - \int_a^b W(s) \left( h(s) + W^*(s)(P_{G_k})^*R_1^{-1}P_{G_k} \left( d - \int_a^b W(s)h(s)ds \right) \right) ,
\]

which we can rewrite in the form

\[
Gx_0 = \left( I_n - R_2(P_{G_k})^*R_1^{-1}P_{G_k} \right) \left( d - \int_a^b W(s)h(s)ds \right) .
\] (13)

Multiplying the right side of this equation from the left by matrix \( P_{G_k} \) we will obtain the zero vector. It means that system (13) is solvable and (because \( \text{rank} G = n - k \) it has \( k \)-parametric solution of the form [2, 4]

\[
x_0 = P_{G_k} \xi + G^+ \left( I_n - R_2(P_{G_k})^*R_1^{-1}P_{G_k} \right) \left( d - \int_a^b W(s)h(s)ds \right) ,
\] (14)

where \( G^+ \) is a unique Moore-Penrose generalized inverse \( (n \times n) \)-matrix [2, 4, 8, 9], \( \xi \in R^k \) is an arbitrary vector. Substituting \( x_0 \) of the form (14) into (12) we obtain the general \( k \)-parametric solution of the BVP (2), (8):

\[
x(t, x_0) = x(t, \xi)
\]

\[
= \Omega_t^a P_{G_k} \xi + \Omega_t^a G^+ \left( I_n - R_2(P_{G_k})^*R_1^{-1}P_{G_k} \right) \left( d - \int_a^b W(s)h(s)ds \right)
\]

\[
+ \int_a^t \Omega_t^a \left( h(s) + W^*(s)(P_{G_k})^*R_1^{-1}P_{G_k} \left( d - \int_a^b W(s)h(s)ds \right) \right) ds .
\]
Finally we can rewrite it in the form
\[ x(t, \xi) = \Omega(t) + \Omega(t) \left( G^+ + (R(t) - G^+ R_2) P_{G}^* R_1^{-1} P_{G}^* \right) d + \int_a^b L(t, s) h(s) ds, \]
where
\[ R(t) = \int_a^t \Omega(s) W^*(s) ds, \]
and
\[ L(t, s) = \begin{cases} \Omega(t) - \Omega(t) \left( G^+ + (R(t) - G^+ R_2) P_{G}^* R_1^{-1} P_{G}^* \right) W(s), & 0 \leq s \leq t \leq b, \\ -\Omega(t) \left( G^+ + (R(t) - G^+ R_2) P_{G}^* R_1^{-1} P_{G}^* \right) W(s), & 0 \leq t < s \leq b. \end{cases} \]

The lemma is proved. \□

3. The numerical-analytic method for investigating nonlinear differential systems with linear boundary condition

Now let us discuss the problem of existence and approximate constructing of the solutions of nonlinear differential systems
\[ \frac{dx}{dt} = A(t) x + f(t, x), \quad x, f \in \mathbb{R}^n, \tag{15} \]
which fulfills the additional condition (2). We will consider a critical case i.e. the case when the condition A is fulfilled.

We suppose that on \( \Omega = [a, b] \times D \), where \( D \subset \mathbb{R}^n \) is a closed and bounded domain, the following conditions are hold:

- **B)** matrix \( A(t) : [a, b] \rightarrow \mathbb{R}^{n \times n} \) and function \( f : \Omega \rightarrow \mathbb{R}^n \) are continuous for \( t \) and the following inequalities are hold:
  \[ |f(t, x)| \leq m(t), \quad |f(t, x') - f(t, x'')| \leq K(t)|x' - x''|, \]
  where vector \( m(t) \) and matrix \( K(t) \) are continuous and their components are nonnegative;

- **C)** the domain \( D_0 \equiv \{ \xi \in \mathbb{R}^k \mid B(x_0(t, \xi), \beta) \subseteq D \} \) is not empty:
  \[ D_0 \neq \emptyset, \]
  
  where
  \[ \beta = \max_{t \in [a, b]} \left( \left| \Omega(t) \left( G^+ + (R(t) - G^+ R_2) P_{G}^* R_1^{-1} P_{G}^* \right) d + \int_a^b L(t, s) m(s) ds \right| \right) \]
and \( B(y, \rho) = \{ x \in \mathbb{R}^n : |x - y| \leq \rho \} \) for fixed \( y, \rho \in \mathbb{R}^n \);

- **D)** maximum eigenvalue of the following matrix \( Q \) is less then one:
  \[ Q = \max_{t \in [a, b]} \int_a^b |L(t, s)| K(s) ds. \]
The notation $|\cdot|$ means the absolute value: $|x| = \text{col}(|x_1|, \ldots, |x_n|)$, where $x \in \mathbb{R}^n$ and $|A| = |A_{ij}|_{i,j=1}^n$. The inequalities are meant componentwise.

Next lemmas give us necessary and sufficient conditions for existing solutions of the BVP (15), (2).

**Lemma 3.1** ([10]). Let the linear homogeneous BVP (7) has $k, 1 \leq k \leq n$ linearly independent solutions and the vector-function $\varphi(t) \in C^1([a, b], \mathbb{R}^n)$ is a solution of the BVP (15), (2) with the initial value

$$\varphi(a) = \varphi_0 := P_{G_k}^* \xi + G^+ \left( d - b \int_a^t W(s)f(s, \varphi(s))ds \right).$$

(16)

Then $\varphi$ is a solution of the system of equations

$$x(t) = \Omega^t_a P_{G_k} \xi$$

$$+ \Omega^t_a \left( G^+ + (R(t)-G^+R_2)(P_{G_k}^*)^* R_1^{-1} P_{G_k}^* \right) d + b \int_a^t L(t, s)f(s, x(s))ds,$$  

(17)

$$P_{G_k} \left( d - b \int_a^t W(s)f(s, x(s))ds \right) = 0.$$  

(18)

**Lemma 3.2.** Let the linear homogeneous BVP (7) has $k, 1 \leq k \leq n$ linearly independent solutions. Then:

1) if the vector-function $\varphi(t)$ satisfies the equation (17) then $\varphi(t) \in C^1([a, b], \mathbb{R}^n)$ and $\varphi(t)$ satisfies the boundary condition (2);

2) if furthermore function $\varphi(t) = \varphi(t, \xi^*)$ satisfies the equation (18), then $\varphi(t) = \varphi(t, \xi^*)$ is a solution of the BVP (15), (2) with the initial value (16).

**Proof.** Let us assume that $\varphi(t)$ satisfies the equation (17). Then $\varphi(t) \in C^1([a, b], \mathbb{R}^n)$ and the identity

$$\varphi(t) = \Omega_a^t P_{G_k} \xi + \Omega_a^t G^+ \left( I_n - R_2(P_{G_k}^*)^* R_1^{-1} P_{G_k}^* \right) \left( d - b \int_a^t W(s)f(s, \varphi(s))ds \right)$$

$$+ b \int_a^t \Omega_a^t f(s, \varphi(s))ds$$

$$+ b \int_a^t \Omega_a^t W^*(s)ds(P_{G_k}^*)^* R_1^{-1} P_{G_k}^* \left( d - b \int_a^t W(s)f(s, \varphi(s))ds \right).$$

(19)
is fulfilled. Substituting $\varphi(t)$ of the form (19) into the boundary condition (2) and taking into the consideration (6), (11) we obtain

$$d - B_1\varphi(0) - B_2\varphi(b) - \int_a^b [dB(t)]\varphi(t)$$

$$= d - \left(B_1 + B_2\Omega_n^b - \int_a^b [dB(t)]\Omega_n^t\right)$$

$$\times \left(P_G\xi + G^+ \left(I_n - R_2(P_G^*)^*R_1^{-1}P_G^* \right) \right) \left(d - \int_a^b W(s)f(s,\varphi(s))ds \right)$$

$$- \left(B_2 \int_a^b \Omega_n^bf(s,\varphi(s))ds + \int_a^b [dB(t)]\int_a^t \Omega_n^t f(s,\varphi(s))ds \right)$$

$$- \left(B_2 \int_a^b \Omega_n^bW^*(s)ds + \int_a^b [dB(t)]\int_a^t \Omega_n^t W^*(s)ds \right)$$

$$\times \left(P_G^*R_1^{-1}P_G^* \right) \left(d - \int_a^b W(s)f(s,\varphi(s))ds \right)$$

$$= d - G^+ \left(I_n - R_2(P_G^*)^*R_1^{-1}P_G^* \right) \left(d - \int_a^b W(s)f(s,\varphi(s))ds \right)$$

$$- \int_a^b W(s)f(s,\varphi(s))ds - R_2(P_G^*)^*R_1^{-1}P_G^* \left(d - \int_a^b W(s)f(s,\varphi(s))ds \right)$$

$$= (I_n - G^+) \left(I_n - R_2(P_G^*)^*R_1^{-1}P_G^* \right) \left(d - \int_a^b W(s)f(s,\varphi(s))ds \right)$$

$$= P_G^* \left(I_n - R_2(P_G^*)^*R_1^{-1}P_G^* \right) \left(d - \int_a^b W(s)f(s,\varphi(s))ds \right)$$

$$= \left(P_G^* - P_G^*R_2(P_G^*)^*R_1^{-1}P_G^* \right) \left(d - \int_a^b W(s)f(s,\varphi(s))ds \right) = 0.$$

Therefore $\varphi(t)$ satisfies the boundary condition (2). If furthermore (18) holds, then from (19) it follows that $\varphi(t)$ satisfies the identity

$$\varphi(t) \equiv \Omega_n^bP_G\xi + \Omega_n^bG^+ \left(d - \int_a^b W(s)f(s,\varphi(s))ds \right) + \int_a^t \Omega_n^t f(s,\varphi(s))ds, \quad (20)$$

i.e. $\varphi(t)$ is a solution of the system (15) with the initial value (16).

Thus the lemma is proved.
For investigating the problem of existence and approximate constructing of the solutions of the BVP (15), (2) we consider the $k$-parametric sequence of functions given by the formula

$$x_m(t, \xi) = x_0(t, \xi) + \Omega_a^t \left( G^+ + (R(t) - G^+ R_2)(P_{G_1})^* R_1^{-1} P_{G_1} \right) d + \int_a^b L(t, s) f(s, x_{m-1}(s, \xi)) ds, \quad x_0(t, \xi) = \Omega_a^t P_{G_1} \xi, \quad \xi \in \mathbb{R}^k, \ m = 1, 2, \ldots. \ (21)$$

It can be easily verified that each of this functions satisfies the boundary condition (2). Let us now establish the main result of this paper.

**Theorem 3.3.** Assume that for the BVP (2), (15) the conditions A—D are hold. Then:

1) the sequence of functions (21) converges as $m \to \infty$, uniformly in $(t, \xi) \in [a, b] \times D_0$ to the limit function $x^*(t, \xi)$ and the following error estimate holds

$$|x^*(t, \xi) - x_m(t, \xi)| \leq (E - Q)^{-1} Q^m \beta; \ (22)$$

2) the limit function $x^*(t) = x^*(t, \xi^*)$ is a solution of BVP (2), (15) if and only if $\xi^*$ is a solution of the determining equation $\Delta(\xi) = 0$, where

$$\Delta(\xi) \stackrel{def}{=} P_{G_1} \left( d - \int_a^b W(s)f(s, x^*(s, \xi)) ds \right) \ (23)$$

and its initial value is

$$x^*(a) = P_{G_1} \xi^* + G^+ \left( d - \int_a^b W(s)f(s, x^*(s, \xi)) ds \right). \ (24)$$

**Proof.** Considering the difference

$$|x_1(t, \xi) - x_0(t, \xi)| \leq \left| \Omega_a^t \left( G^+ + (R(t) - G^+ R_2)(P_{G_1})^* R_1^{-1} P_{G_1} \right) d \right|$$

$$+ \int_a^b |L(t, s)f(s, x_0(s, \xi))| ds \leq \beta,$$

we see that $x_1(t, \xi) \in D$. It can be shown by induction that $x_m(t, \xi) \in D$ for all $\xi \in D_0, \ m \in \mathbb{N}$. Furthermore, from the Lipschitz condition we obtain the estimates:

$$|x_{m+1}(t, \xi) - x_m(t, \xi)| \leq \int_a^b |L(t, s)| |f(s, x_m(s, \xi)) - f(s, x_{m-1}(s, \xi))| ds$$

$$\leq Q |x_m(t, \xi) - x_{m-1}(t, \xi)| \leq Q^2 |x_{m-1}(t, \xi) - x_{m-2}(t, \xi)| \leq \cdots \leq Q^m \beta,$$
and thus

\[ |x_{m+j}(t, \xi) - x_m(t, \xi)| \leq \sum_{i=0}^{j-1} |x_{m+i+1}(t, \xi) - x_{m+i}(t, \xi)| \]

\[ \leq \sum_{i=0}^{j-1} Q^{m+i} |x_1(t, \xi) - x_0(t, \xi)| \leq \sum_{i=0}^{j-1} Q^{m+i} \beta. \quad (25) \]

From (25) and the condition \( D \) it follows that \( x_m(t, \xi) \) is a Cauchy sequence. Therefore it uniformly converges to a continuous function \( x^*(t, \xi) \). Passing to the limit as \( j \to \infty \) in (25), we prove the error estimate (22). Since all functions satisfy the boundary condition (2), we conclude that so does the limit function \( x^*(t, \xi) \) for arbitrary \( \xi \in \mathbb{R}^k \). Taking the limit as \( m \to \infty \) we get that \( x^*(t, \xi) \) satisfies the equation (17). According to the lemmas 3.1, 3.2, the limit function \( x^*(t, \xi^*) \) is a solution of the BVP (15), (2) if and only if the condition \( \Delta(\xi^*) = 0 \) is fulfilled and in this case its initial value is of the form (24).

Thus the theorem is proved. \( \Box \)

The next statement gives us the satisfactory conditions for the existence of the solution of the boundary value problem (2), (15). This conditions are based upon the properties of the approximations \( x_m(t, \xi) \) and not upon those of the limit function \( x^*(t, \xi) \).

**Theorem 3.4.** Assume that for the BVP (2), (15) the conditions A—D are hold and furthermore:

1) there exists closed convex subset \( D_1 \subset D_0 \subset \mathbb{R}^k \) such that for some fixed \( m \in \mathbb{N} \) the approximate equation

\[ \Delta_m(\xi) \overset{d}{=} P G_k \left( d - \int_a^b W(s) f(s, x_m(s, \xi)) ds \right) = 0 \quad (26) \]

possesses only one isolated solution \( \xi = \xi_{0m} \) of non-zero index;

2) on the boundary \( \partial D_1 \) of the set \( D_1 \) the condition

\[ \inf_{\xi \in \partial D_1} |\Delta_m(\xi)| > Q_1 (I_n - Q)^{-1} Q^m \beta, \quad (27) \]

is satisfied, where \( Q_1 = \int_a^b |P G_k W(s)| K(s) ds \).

Then there exists a solution \( x = x^*(t) = x^*(t, \xi^*) \) of the boundary value problem (2), (15), where \( \xi^* \in D_1 \).

**Proof.** We introduce the continuous for \( \xi \in \partial D \) and \( \theta \in [a, b] \) vector field family

\[ \Delta(\theta, \xi) = \Delta_m(\xi) + \theta(\Delta(\xi) - \Delta_m(\xi)), \quad 0 \leq \theta \leq 1, \]

connecting the vector fields \( \Delta(0, \xi) = \Delta_m(\xi) \) and \( \Delta(1, \xi) = \Delta(\xi) \). Let us assume that there exists \( \theta_0 \in [0, 1] \) such that \( \Delta(\theta_0, \xi) = 0 \). Then

\[ \Delta_m(\xi) = -\theta_0(\Delta(\xi) - \Delta_m(\xi)). \quad (28) \]
From (22), (23), (26) and the Lipschitz condition we have

\[ |\Delta(\xi) - \Delta_m(\xi)| \leq \int_a^b |P_G^* W(s)| \cdot |f(s, x^*(s, \xi)) - f(s, x_m(s, \xi))| ds \leq \]

\[ \leq \int_a^b |P_G^* W(s)| K(s) |x^*(s, \xi) - x_m(s, \xi)| ds \leq Q_1 (I_n - Q)^{-1} Q_m \beta. \]

But in this case from (28) we obtain the inequality

\[ |\Delta_m(\xi)| \leq |\Delta(\xi) - \Delta_m(\xi)| \leq Q_1 (I_n - Q)^{-1} Q_m \beta, \]

which contradict the condition (27). It means that the field family \(\Delta(\theta, \xi)\) does not assume the value zero on \(\partial D_1\), therefore vector fields \(\Delta(\xi)\) and \(\Delta_m(\xi)\) are homotopic. It means that the rotation of vector field \(\Delta(\xi)\) on the boundary \(\partial D_1\) is also non-zero and consequently \(\Delta(\xi)\) assumes the value zero at least in one point \(\xi = \xi^* \in D_1\). Thus the theorem is proved. \(\square\)

References


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AN ALGORITHM TO SOLVE SOME TYPES OF FIRST ORDER DIFFERENCE EQUATIONS

J. LAITOCHOVÁ

Abstract. The classical theory of first-order linear functional and difference equations is obtained as a special case of the theory developed here for the functional equation model

\[ f \circ \Phi - Af = B. \]

The setting is a special space \( S \) of continuous strictly monotonic functions, where \( \circ \) is a group multiplication defined on \( S \). Iterative solution formulas are given in the two cases when \( A, B \) are constant and when \( A, B \) depend on \( x \).

Introduction

The difference equation

\[ f(x + 1) - 2f(x) = 5 \]  

and the functional equation

\[ f\left(\frac{1}{1+x^2} - \frac{1}{2} + x\right) - xf(x) = 0 \]

are instances of equations represented in this article by a unifying abstract model

\[ f \circ \Phi - Af = B \]

where \( \circ \) is a group multiplication defined on monotonic functions, defined in a later section.

Within the abstract framework, solution formulas for (3) can be developed (see [5]) and the structure of solutions can be documented (see [6, 7]). The formulas apply equally to difference equations like (1) and functional equations like (2).

Examples are given in the last section of the paper.

Some basic references for the topic of the paper are [1, 2, 8].

1. First order difference equations in the space \( S \)

Denote by \( C_0(J) \) the set of continuous functions on the interval \( J = (-\infty, \infty) \), \( Z \) the set of integers and \( R \) the set of real numbers.

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The setting for the model of functional and difference equations is in the space $S$:

**Definition 1.1.** Let $a, b$ be real or extended numbers, $a < b$. The set of all functions $f$ defined on the interval $J = (-\infty, \infty)$ which satisfy
1. $f \in C_0(J)$,
2. $f$ maps one-to-one the interval $J$ on the interval $(a, b)$,
will be denoted by the symbol $S$ and called a space of strictly monotonic functions.

**Definition 1.2.** An arbitrarily chosen increasing function $X = X(x), X \in S$, will be called a canonical function in $S$. The inverse to the canonical function $X$ will be denoted by $X^*$.

**Remark 1.3.** In the classical theory of difference equations, set $S$ is the space of monotonic functions defined on $J$ and the canonical function is the identity function $X(x) = x$.

**Definition 1.4.** Let $\alpha, \beta \in S$. Let $X^*$ be the inverse to the canonical function $X \in S$. The composite function $\gamma(x) = \alpha(X^*(\beta(x)))$ is defined on $J$ and it will be called a product of functions $\alpha, \beta \in S$ in the class $S$, denoted as $\gamma = \alpha \circ \beta$.

**Remark 1.5.** In the classical theory of difference equations we use composition of functions for the operation $\circ$.

**Definition 1.6.** Let $X \in S$ be a canonical function. Let $\Phi \in S$. The iterates of a function $\Phi$ in $S$ are given by
\begin{align*}
\Phi^0(x) &= X(x), \\
\Phi^{n+1}(x) &= (\Phi \circ \Phi^n)(x), & x \in J, \ n = 0, 1, 2, \ldots, \\
\Phi^{n-1}(x) &= (\Phi^{-1} \circ \Phi^n)(x), & x \in J, \ n = 0, -1, -2, \ldots,
\end{align*}
where $\Phi^{-1} = \hat{\Phi}$ is the inverse element to the element $\Phi$ in $S$ according to the multiplication $\circ$.

**Remark 1.7.** In the classical theory of difference equations we have $X(x) = x$ and
\begin{align*}
\Phi^1(x) &= \Phi(x), & \Phi^{n+1}(x) &= \Phi(\Phi^n(x)), & n = 1, 2, \ldots.
\end{align*}

**Definition 1.8.** Let $\Phi \in S$. A function $p = p(x), p \in C_0(J)$, is called automorphic over $\Phi$ if
\[ p(x) = (p \circ \Phi)(x) \quad \text{for} \quad x \in J. \]
Definition 1.9. Let \( f \in C_0(J), \Phi, X \in S \). Let \( X \) be a canonical function. The \textbf{difference operator} of a function \( f \) relative to \( \Phi \) is denoted by \( \Delta_{\Phi} f(x) \) and defined by the equation

\[
\Delta_{\Phi} f(x) = (f \circ \Phi)(x) - f(x).
\]

Remark 1.10. In the classical theory of difference equations we have

\[
\Delta f(x) = f(x + h) - f(x), \quad h > 0.
\]

It is easy to prove that the \( n \)-th difference of a function \( f(x) \) relative to \( \Phi \) at \( x \in J \) is

\[
\Delta_n^{\Phi} f(x) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (f \circ \Phi^i)(x).
\]

The new model equation, that is the \textbf{linear functional (difference) equation of \( k \)-th order in} \( S \), is given by the following equation

\[
p_k(x) \Delta_k^{\Phi} f(x) + p_{k-1}(x) \Delta_{k-1}^{\Phi} f(x) + \cdots + p_0(x) f(x) = Q(x). \tag{8}
\]

Equation (8) can be re-written in the form

\[
a_k(x) (f \circ \Phi^k)(x) + \cdots + a_0(x) (f \circ \Phi^0)(x) = Q(x), \tag{9}
\]

where \( a_i, Q \in C_0(J), i = 0, 1, \ldots, k, x \in J \). We look for a solution \( f \in C_0(J) \) which satisfies equation (9) identically on \( J \) in the case of the functional equation or \( x \in \{ x_{\mu} \}_{\mu=0}^{\infty} \), where \( x_{\mu} = X^\ast(\Phi^\mu(x_0)), x_0 \in J \) in the case of the difference equation.

The general theory of linear \( k \)th order functional and difference equations in \( S \) was developed in [6, 7].

The linear homogeneous functional (difference) equation of \( k \)th order with \textbf{constant coefficients} is of the form

\[
a_k \left( f \circ \Phi^k \right)(x) + a_{k-1} \left( f \circ \Phi^{k-1} \right)(x) + \cdots + a_0 \left( f \circ \Phi^0 \right)(x) = 0, \tag{10}
\]

where \( a_k = 1 \) and \( a_0, \ldots, a_{k-1} \) are constants.

1.1. \textbf{Abel functional equations in} \( S \)

In the space \( S \) we define the \textbf{generalized equation of Abel type in} \( S \) as

\[
(\alpha \circ \Phi)(x) = (\Phi \circ \alpha)(x),
\]

where \( X, \Phi \) are given and \( \alpha \) is the unknown function. If \( X(x) = x \), then it specializes to the \textbf{Abel functional equation in} \( S \),

\[
(\alpha \circ \Phi)(x) = X(x + 1) \circ \alpha(x).
\]

Remark 1.11. In the classical theory of difference equations, the Abel functional equation reduces to

\[
\alpha(\Phi(x)) = \alpha(x) + 1.
\]

Definition 1.12. A fixed point \( \bar{x} \) of \( \Phi \) is defined by the equation

\[
\Phi(\bar{x}) = X(\bar{x}),
\]

where \( X(x) \) is a canonical function in \( S \).
Given an increasing function $\Phi$, possibly having fixed points in its domain $(a, b)$, a group-theoretic iterative explicit construction is given in reference [4] for infinitely many solutions $\alpha$ of the Abel functional equation in $S$. Each solution $\alpha$ is infinite at fixed points of $\Phi$ and otherwise monotonic.

**Theorem 1.13.** Let equation (10) be given and denote by $X^*$ the inverse function of canonical function $X$. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be simple positive roots of the characteristic equation
\[
a_k \lambda^k + a_{k-1} \lambda^{k-1} + \cdots + a_1 \lambda + a_0 = 0. \tag{11}\]
Let $\alpha$ be a solution of the associated Abel functional equation
\[
(\alpha \circ \Phi)(x) = X(x + 1) \circ \alpha(x). \tag{12}\]
Then the functions
\[
f_1(x) = \lambda_1 X^*(\alpha(x)), \quad f_2(x) = \lambda_2 X^*(\alpha(x)), \quad \ldots, \quad f_k(x) = \lambda_k X^*(\alpha(x)) \tag{13}\]
are linearly independent solutions of equation (10).

For a proof see [3].

In the following subsections, we cite some special results from [5], where the construction appears.

1.2. First order linear functional and difference equations with constant coefficients and a constant right side

Consider the first order linear functional equation
\[
f \circ \Phi(x) = Af(x) + B, \]
where $A, B$ are constants, $A \neq 0$. The following solution formulas are proved in reference [5].

**Theorem 1.14.** Assume $A = 1$ and $B$ is a constant. The general solution of the nonhomogeneous equation $(f \circ \Phi)(x) = Af(x) + B$ is a sum of the general solution of the homogeneous equation $(F \circ \Phi)(x) = AF(x)$ and a particular solution of the nonhomogeneous equation $(f \circ \Phi)(x) = Af(x) + B$. That is,
\[
f(x) = C(x)a(x) + \Delta_{\Phi}^{-1}B, \]
where $a(x), C(x)$ are automorphic over $\Phi$ and $B \in \mathbb{R}$.

**Theorem 1.15.** Assume $A \neq 1, A > 0, B$ are constants. Then each solution of the nonhomogeneous equation $(f \circ \Phi)(x) = Af(x) + B$ is a sum of the general solution of homogeneous equation $(F \circ \Phi)(x) = AF(x)$ and a particular solution of the nonhomogeneous equation $(f \circ \Phi)(x) = Af(x) + B$. Therefore,
\[
f(x) = C(x)A^{X^*(\alpha(x))} + \frac{B}{1-A}, \]
where $C(x)$ is automorphic over $\Phi$ and $\alpha(x)$ is a solution of the Abel equation $(\alpha \circ \Phi)(x) = X(x + 1) \circ \alpha(x)$. 
1.3. First order linear functional and difference equations with nonconstant coefficients

Consider for continuous functions $A, B$ on $\langle x_0, \infty \rangle$, $A(x)$ never zero, the equation
 \[(f \circ \Phi)(x) - A(x)(f \circ \Phi^0)(x) = B(x)\].

**Theorem 1.16.** The general solution $f$ of the first order nonhomogeneous equation $(f \circ \Phi)(x) - A(x)f = B(x)$ with continuous nonconstant coefficients $A, B$ on $\langle x_0, \infty \rangle$ is a continuous function
 \[f(x) = \left( A \circ \Phi^{-1} \right)(x) \left( A \circ \Phi^{-2} \right)(x) \cdots \left( A \circ \Phi^{-n-1} \right)(x) G(x) \]
where
 \[G(x) = \Delta_{\Phi}^{-1} \left[ \frac{B(x)}{\left( A \circ \Phi^0 \right)(x) \left( A \circ \Phi^{-1} \right)(x) \cdots \left( A \circ \Phi^{-n-1} \right)(x)} \right] + k(x) \]
and $k(x)$ is automorphic over $\Phi$.

A proof can be found in [5].

1.4. Examples

**Example 1.17.** Define $X(x) = x$ and consider the Abel functional equation
 \[\alpha(\Phi(x)) = \alpha(x) + 1, \quad \text{where} \quad \Phi(x) = \frac{1}{1+x^2} - \frac{1}{2} + x.\]

Graph possible solutions of the equation.

**Solution.** The natural domain of $\Phi$ is the real line $(-\infty, \infty)$, but we can consider other possible domains $(-\infty, -1), (-1, 1)$ and $(1, \infty)$ equally natural. This is because the fixed points $x = \pm 1$ of $\Phi$, which satisfy $\Phi(-1) = -1, \Phi(1) = 1$, prevent $x = \pm 1$ from being used in the given equation. We get a solution $\alpha$ on $J$. 

![Figure 1](image-url)
which is defined piecewise on three intervals. A graphic showing one possible global iterative solution \( \alpha(x) \) appears in Figure 1. The fixed points separate regions of increase and decrease of \( \alpha \). The black and grey segments of the solution \( \alpha \) represent the iterative steps used to produce the graphic. Generally, infinitely many solutions are required to fully represent \( \alpha \). The tick marks on the \( x \)-axis show the intervals on which the iterations are made.

The Abel functional equation \( \alpha(\Phi(x)) = \alpha(x) + 1 \), where \( \Phi(x) = \frac{1}{1+x^2} - \frac{1}{2} + x \), can be considered a difference equation, in which case the orbit of a point \( x_0 \) generates the iterative solution. This solution \( y = \alpha(x) \) is a discrete sequence of points in the plane, depicted as the open circles in Figure 2. □

![Figure 2. A discrete iterative solution \( y = \alpha(x) \) for the difference equation \( \alpha(\Phi(x)) = \alpha(x) + 1 \), where \( \Phi(x) = \frac{1}{1+x^2} - \frac{1}{2} + x \).](image)

**Example 1.18.** Consider the space \( S \), with \( J \) replaced by \([0, \infty)\). The canonical function \( X \) is defined by \( X(x) = e^x \). We find a solution \( f \) of the nonhomogeneous equation

\[
(f \circ \Phi)(x) = (f(x)) + 1,
\]

where

\[
\Phi(x) = \begin{cases} 
\frac{1}{1+x^2}, & x < -1, \\
x + 1.5, & x \geq -1.
\end{cases}
\]

**Solution.** The problem can be converted to

\[
f(F(x)) = G(f(x)),
\]

where \( G(u) = 1 + u \) and \( F(x) = X^{-1}(\Phi(x)) = \begin{cases} 
\ln(x + 1.5), & x > -1, \\
\ln \frac{1}{1+x^2}, & x \leq -1.
\end{cases}\)

Figure 3 shows a possible solution \( f \). □

**Example 1.19.** For the space \( S \) and canonical function \( X(x) = x \), consider the nonhomogeneous equation with nonconstant coefficients

\[
f \circ \Phi(x) = (2 + \sin(6\pi x))f(x) + 1.
\]
We solve for $f$ when
\[ \Phi(x) = \frac{1}{1+x^2} - \frac{1}{2} + x. \]

**Solution.** The problem can be converted to
\[ f(F(x)) = A(x)f(x) + B(x), \]
where $A(x) = 2 + \sin(6\pi x)$, $B(x) = 1$ and $F(x) = \frac{1}{1+x^2} - \frac{1}{2} + x$.

Because $F$ has two fixed points $x = 1$ and $x = -1$, iterative methods can be applied on the three intervals between fixed points; see Figure 4.

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**Example 1.20.** Consider the problem
\[ (f \circ \Phi)(x) = f(x) + (0.1 + \sin(6\pi x)). \]
The canonical function $X$ is $X(x) = x$. We solve for $f$ when
\[ \Phi(x) = \frac{1}{1+x^2} - \frac{1}{2} + x. \]
Solution. The problem can be converted to
\[ f(F(x)) = A(x)f(x) + B(x), \]
where \( A(x) = 1, \ B(x) = 0.1 + \sin(6\pi x) \) and \( F(x) = \frac{1}{1+2x} - \frac{1}{2} + x. \)

The function \( F \) is continuously differentiable and \( F'(x) > 0 \). Because \( F \) has two fixed points \( x = 1 \) and \( x = -1 \), iterative methods can be applied on the three intervals between fixed points. A solution \( f(x) \) is displayed in Figure 5.

\[ \text{Figure 5. A solution for } f(\Phi(x)) = f(x) + (0.1 + \sin(6\pi x)). \]

References

THE NUMERICAL MODEL OF FORECASTING ALUMINIUM PRICES BY USING TWO INITIAL VALUES

MARCELA LASCSÁKOVÁ

Abstract. In mathematical models forecasting the prices on the commodity exchanges statistical methods are usually used. In the paper we derive the numerical model based on the numerical solution of the Cauchy initial problem for the 1st order ordinary differential equations to prognose the prices of Aluminium on the London Metal Exchange. When forecasting monthly average prices, we compare the accuracy of the prognoses acquired either in the direct way or as the arithmetic mean of daily prognoses. The advantages of the studied types of forecasting during different movements of Aluminium prices are analyzed.

INTRODUCTION

One of the most important factors determining the prices of the non-ferrous metals is the impact of the London Metal Exchange (LME). It is the world’s premier non-ferrous metals market. The origins of LME can be traced back as far as the opening of the Royal Exchange in London in 1571. Although there are other commodity exchanges where metals are traded (for example NYMEX in the USA or SIMEX in Singapore) the majority of producers and businessmen respect just the official prices daily closed on LME.

Observing trends and forecasting the movements of metal prices is still a current problem. There are a lot of approaches to forecasting the price movements. Some of them are based on mathematical models. Forecasting the prices on the commodity exchanges often uses statistical methods that need to process a large number of historical market data. The amount of needed market data can sometimes be a disadvantage. In such cases other mathematical methods are required.

We have decided to use numerical methods. Their advantage is that much less market data is needed in comparison with the statistical models. Our numerical model for forecasting prices is based on the numerical solution of the Cauchy initial problem for the 1st order ordinary differential equations.

Let us consider the Cauchy initial problem in the form

\[ y' = f(x, y), \quad y(x_0) = y_0. \]  

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We assume that there exists just one solution \( y(x) \) of the problem (1) in the interval \( (\alpha, \beta) \), which has an exponential character. Based on this assumption we shall consider the Cauchy initial problem in the form

\[
y' = a_1 y, \quad y(x_0) = y_0.
\]  

(2)

The particular solution of the problem (2) is \( y = a_0 e^{kx} \), where \( a_0 = y_0 e^{-a_1 x_0}, \; k = a_1 \).

In our prognostic model we came out of the Aluminium prices presented on LME. We dealt with the monthly averages of the daily closing Aluminium prices “Cash Seller&Settlement price” in the period from December 2002 to June 2006. We obtained the market data from the official web page of the London Metal Exchange [3]. The course of the Aluminium prices on LME (in US $ per tonne) in the observing period is presented in Figure 1.

![Figure 1. The course of the Aluminium prices on LME in years 2003–2006.](image)

As we can see in Figure 1 the course of the Aluminium prices in the considered period changes dramatically.

1. **Mathematical model**

We shall consider the Cauchy initial problem in the form (2)

\[
y' = a_1 y
\]
with the initial condition
\[ y(x_0) = Y_0, \]
where \([x_0, Y_0]\) are the known values (\(x_0\) is the order of month, let \(x_0 = 0\) and \(Y_0\) is the Aluminium price (stock exchange) on LME in the month \(x_0\)).

To determine the value of unknown coefficient \(a_1\), the second known point \([x_1, Y_1]\) is used, where \(x_1 = 1\) and \(Y_1\) is the Aluminium price on LME in the month \(x_1\). That means \([x_1, Y_1]\) are the values corresponding to the next month in comparison with those of \([x_0, Y_0]\).

Substituting the point \([x_1, Y_1]\) to the particular solution of the problem (2) we have
\[ Y_1 = Y_0 e^{a_1(x_1 - x_0)}. \]

After some manipulations we obtain the formula of unknown coefficient \(a_1\)
\[ a_1 = \frac{1}{x_1 - x_0} \ln \left( \frac{Y_1}{Y_0} \right). \]

Now we can substitute \(a_1\) to the Cauchy initial problem (2) and we acquire
\[ y' = \frac{1}{x_1 - x_0} \ln \left( \frac{Y_1}{Y_0} \right) \cdot y, \quad y(x_1) = Y_1. \]

Generalizing the previous principle we can get the Cauchy initial problem in the point \(x_i\) in the following form
\[ y' = \frac{1}{x_i - x_{i-1}} \ln \left( \frac{Y_i}{Y_{i-1}} \right) \cdot y, \quad y(x_i) = Y_i, \quad i = 1, 2, 3, \ldots \quad (3) \]

The unknown values of Aluminium prices are forecasted by the numerical solution of the Cauchy initial problem (3) using the method of exponential forecasting. This numerical method is based on the exponential approximation of the solution of the Cauchy initial problem for the 1st order ordinary differential equations [1].

The method is using the following numerical formulae
\[ x_{i+1} = x_i + h, \quad y_{i+1} = y_i + bh + Qe^{vx_i} \left( e^{vh} - 1 \right), \]
for \(i = 1, 2, \ldots\), where \(h = x_{i+1} - x_i\) is the step of constant size.

The unknown coefficients are calculated by means of these formulae
\[
\begin{align*}
v &= \frac{f''(x_i, y_i)}{f'(x_i, y_i)}, \\
Q &= \frac{f'(x_i, y_i) - f''(x_i, y_i)}{(1 - v)e^{vx_i}}, \\
b &= f(x_i, y_i) - \frac{f'(x_i, y_i)}{v}.
\end{align*}
\]
If we consider the Cauchy initial problem \( (3) \), the function \( f(x_i, y_i) \) has the form
\[
f(x_i, y_i) = a_1 y_i,
\]
and then
\[
f'(x_i, y_i) = a_1 y_i',
\]
\[
f''(x_i, y_i) = a_1 y_i'',
\]
where
\[
a_1 = \frac{1}{x_i - x_{i-1}} \ln(Y_i/Y_{i-1}).
\]

Using two known initial values \([x_{i-1}, Y_{i-1}]\) and \([x_i, Y_i]\) we calculate the prognosis \( y_{i+1} \) in the month \( x_{i+1} \), \( i = 1, 2, 3, \ldots \). In this way, using the known Aluminium stock exchanges in December 2002 and January 2003, we gradually determine price prognoses in the next months in the period from February 2003 to June 2006. Thus, we obtain the prognoses for 41 months. Each Aluminium price is calculated by using a current form of the Cauchy initial problem \( (3) \).

Two types of forecasting are created:

1. **Monthly forecasting.**
   Using the values \([x_{i-1}, Y_{i-1}]\) and \([x_i, Y_i]\) we directly obtain \([x_{i+1}, y_{i+1}]\) for \( i = 1, 2, 3, \ldots, 40 \).

2. **Daily forecasting.**
   In this case the prognosis \( y_{i+1} \) in the month \( x_{i+1} \) is obtained from the values \([x_{i-1}, Y_{i-1}]\) and \([x_i, Y_i]\) using more partial computations. The interval \( \langle x_i, x_{i+1} \rangle \) of the length \( h = 1 \) month is divided into \( n \) parts, where \( n \) is the number of trading days on LME in the month \( x_{i+1} \). We gain the sequence of points \( x_{i0} = x_i, x_{ij} = x_i + \frac{h}{n} \cdot j, \) for \( j = 1, 2, \ldots, n \), where \( x_{in} = x_{i+1} \). For each point of the subdivision of interval a current form of the Cauchy initial problem \( (3) \), which is solved by the chosen method of exponential forecasting, is created. In this way we obtain the prognoses of Aluminium price on trading days \( y_{ij} \). By computing the arithmetic mean of the daily prognoses we obtain the monthly prognosis of Aluminium price in the month \( x_{i+1} \). Thus, \( y_{i+1} = (\sum_{j=1}^{n} y_{ij})/n \).

This type of forecasting responds more to creating the real monthly averages of the daily closing Aluminium prices on the London Metal Exchange. Consequently, we assume that this type of forecasting gives us more accurate prognoses than monthly forecasting.

The calculated prognoses are compared with the real stock exchanges. We evaluate the difference between them, which is denoted by \( \delta_s = y_s - Y_s \) (prognosis deviation) and the ratio of prognosis deviation from the real price, i.e.
\[
p_s = \frac{\delta_s}{Y_s} \cdot 100%, \quad s = 2, 3, \ldots, 42.
\]

The numerical computations are made by using programs written in Turbo Pascal.
2. Results

2.1. Comparing the accuracy of both monthly forecasting and daily forecasting

We deal with the comparison of the accuracy of the chosen types of forecasting during solving the Cauchy initial problem (3) by the numerical method of exponential forecasting. The forecasting types were compared by means of the following criterions:

- the number of more accurate price prognoses when comparing the two chosen types of forecasting in the observing months,
- the number of critical values of forecasting (it means the prognosis with the absolute percentage error exceeded 10% of the real stock exchange),
- the arithmetic mean of all absolute percentage monthly deviations, \( \sum_{s=2}^{42} |p_s|/41 \).

The obtained results are presented in Table 1.

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Monthly forecasting</th>
<th>Daily forecasting</th>
</tr>
</thead>
<tbody>
<tr>
<td>The number of more accurate price prognoses</td>
<td>13</td>
<td>28</td>
</tr>
<tr>
<td>The number of critical values</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>The arithmetic mean of all absolute percentage monthly deviations</td>
<td>4.65%</td>
<td>3.99%</td>
</tr>
</tbody>
</table>

Table 1. The results of price forecasting by means of the method of exponential forecasting.

As we can see in Table 1 the arithmetic means of all absolute percentage monthly deviations are not different markedly. But we can say that daily forecasting is more accurate. The lower arithmetic mean of all absolute percentage monthly deviations, fewer critical values and more than twice higher number of more accurate prognoses than monthly forecasting point at this fact. Better results of daily forecasting are also confirmed by the distribution of the number of prognoses according to their absolute percentage monthly deviation from the real price, see Table 2.

In Table 2 we can see that by daily forecasting we obtained more prognoses with the lower absolute percentage monthly deviation than by monthly forecasting.

2.2. The analyze of the higher accuracy of daily forecasting

Comparing the values of both types of forecasting and the Aluminium stock exchange in the observing months we have found out that the daily forecasted prognoses are lower when the price increases and they are higher when the price
The absolute percentage monthly deviation forecasting is less than 5% for 27 cases and 30 cases in daily and monthly forecasting, respectively. For percentage deviations between 5% and 7.5%, there are 8 cases in monthly forecasting and 7 cases in daily forecasting. For deviations between 7.5% and 10%, the number of cases is 3 for both forecasting methods. For deviations greater than or equal to 10%, there are 3 cases in monthly forecasting and 1 case in daily forecasting.

<table>
<thead>
<tr>
<th>Absolute Percentage Monthly Deviation</th>
<th>Monthly Forecasting</th>
<th>Daily Forecasting</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 5%</td>
<td>27</td>
<td>30</td>
</tr>
<tr>
<td>(5%, 7.5%)</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>(7.5%, 10%)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>≥ 10%</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2. The distribution of the number of the prognoses according to their absolute percentage monthly deviation from the real price.

decreases in comparison with the prognoses obtained by monthly forecasting. Thereby neither the increase nor the decrease of the daily forecasted prognosis is large. Therefore daily forecasting can be considered slighter than monthly forecasting. The advantages of daily forecasting are especially visible when the price trend is changed. It can be seen in the following situations:

- the trend of the prices is increasing ($Y_{i-1} < Y_i$), but the following prognosed price $Y_{i+1}$ falls,
- the trend of the prices is decreasing ($Y_{i-1} > Y_i$), but the following prognosed price $Y_{i+1}$ increases.

The computed prognosis keeps the trend of two previous initial values (it means these relations hold $Y_{i-1} < Y_i < y_{i+1}, Y_{i-1} > Y_i > y_{i+1}$, respectively). In the increasing trend the prognosis increases, too. The increase of the daily forecasted prognosis $y_{i+1}$ is smaller than the increase of the prognosis obtained by monthly forecasting. Thus, the daily forecasted prognosis is closer to the real stock exchange, which is fallen. In the decreasing trend the prognosis falls, too. The decline of the daily forecasted prognosis $y_{i+1}$ is also smaller than the decline of the monthly forecasted prognosis. Therefore the prognosis obtained by daily forecasting is closer to the real stock exchange in the increase. In the described cases of the variable trends daily forecasting was always more successful than monthly forecasting.

There are also stable trends in the course of the Aluminium prices:

- the trend of the prices is increasing and the next prognosed price increases, too ($Y_{i-1} < Y_i < Y_{i+1}$),
- the trend of the prices is decreasing and the next prognosed price falls, too ($Y_{i-1} > Y_i > Y_{i+1}$).

In these cases both chosen types of forecasting are more accurate under the different circumstances. The success of the forecasting types depends on the intensity of either the increase or the decrease in the three observed months $[x_{i-1}, Y_{i-1}], [x_i, Y_i]$ and $[x_{i+1}, Y_{i+1}]$. If the increase or the decrease in the price is rapid, it
means $|Y_{i+1} - Y_i| > |Y_i - Y_{i-1}|$, then monthly forecasting is more accurate. In the moderate increase or decrease, it means when $|Y_{i+1} - Y_i| < |Y_i - Y_{i-1}|$, daily forecasting is more accurate. It holds for all 20 prognoses computed in the periods of the stable trend. Table 3 shows the number of more accurate prognoses by comparing the mentioned types of forecasting according to the trends of the Aluminium prices.

<table>
<thead>
<tr>
<th>The trend of the prices</th>
<th>Monthly forecasting</th>
<th>Daily forecasting</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stable trend</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>the trend of the prices is increasing and the prognosed price increases</td>
<td>11</td>
<td>6</td>
<td>17</td>
</tr>
<tr>
<td>the trend of the prices is decreasing and the prognosed price falls</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Variable trend</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>the trend of the prices is increasing and the prognosed price falls</td>
<td>0</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>the trend of the prices is decreasing and the prognosed price increases</td>
<td>0</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Total</td>
<td>13</td>
<td>28</td>
<td>41</td>
</tr>
</tbody>
</table>

Table 3. The number of more accurate prognoses by comparing the mentioned types of forecasting according to the trends of the Aluminium prices on LME.

Having the lower number of critical values daily forecasting is more accurate. The lower number of critical values forecasting gains, the more successful it is. From the set of 41 calculated prognoses, only 3 monthly forecasted prognoses (in May 2004 (11.31%), April 2005 (10.18%) and June 2006 (26.08%)) and 1 daily forecasted prognosis (in June 2006 (20.95%)) acquired the absolute percentage error higher than 10% of the real price. (The number in the brackets presents percentage deviation of the prognosis from the real price.)

The cause of the forecasting fail in the above mentioned months is the rapid fall in the prices after the increase. The more rapid change is, the bigger error of the prognosis is. Considering the previous analyze of the accuracy of the forecasting types depending on the trends of the prices, we can assume that daily forecasting is more accurate during the trend of the prices changes. In May 2004 and April 2005, where the change was not so rapid (the decrease in the price in May 2004 presented 6.16% of the price in April 2004 and the decrease in the price in April 2005 was 4.44% of the price in March 2005), by daily forecasting we obtained the price prognoses with the percentage errors less than 10% of the real price (in May 2004 it was 9.03% and in April 2005 the percentage error was 7.52%). But the break in June 2006 was so large (13.42% of the price in May 2006) that daily forecasting was not able to catch the break after the rapid increase, either.
3. Conclusion

To forecast the prices on the commodity exchanges, numerical models need much less market data than statistical models. The presented prognostic numerical model has no problem to catch the rapid increase in the increasing trend, or the rapid fall in the decreasing trend. The changes in the stable trends cause the problems in forecasting. If the change is too large, forecasting does not catch the change and the absolute price error rises over 10% of the stock exchange. Daily forecasting can make forecasting more accurate. It can achieve the trend changes better than monthly forecasting.

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ON ASYMPTOTICS OF CONDITIONALLY OSCILLATORY HALF-LINEAR EQUATIONS

ZUZANA PÁTÍKOVÁ

Abstract. We use a combination of Riccati technique together with the concept of perturbations of a general nonoscillatory half-linear differential equation

\[(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1} \text{sgn } x, \quad p > 1,\]

where \(r, c\) are continuous functions, \(r(t) > 0\), to derive an asymptotic formula for a nonoscillatory solution of the conditionally oscillatory half-linear second order differential equation.

1. Introduction

Qualitative properties of solutions of a general half-linear second order differential equation

\[(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1} \text{sgn } x, \quad p > 1,\]

(1)

where \(t \in [t_0, \infty)\), \(r, c\) are continuous functions and \(r(t) > 0\), have recently been studied by many authors (see e.g. [1, 7]). It is well known that this equation can be classified in the same way as in the theory of linear second order differential equations as oscillatory if its every nontrivial solution has infinitely many zeros tending to infinity and as nonoscillatory otherwise. In the next let us suppose that equation (1) is nonoscillatory and let \(d(t)\) be a positive continuous function. If there exists a positive constant \(\mu_0\) such that equation

\[(r(t)\Phi(x'))' + [c(t) + \mu d(t)]\Phi(x) = 0 \quad \text{for } \mu > \mu_0,\]

(2)

is oscillatory for \(\mu > \mu_0\) and nonoscillatory for \(\mu < \mu_0\), we say that (2) is conditionally oscillatory with a critical constant \(\mu_0\).

Having a nonoscillatory equation (1), authors of [3] constructed a conditionally oscillatory equation by adding a suitable term to the left hand side of (1) in the form

\[(r(t)\Phi(x'))' + \left[\frac{c(t) + \frac{\mu}{h \rho(t)R(t)(\int_{t_0}^{t} R^{-1}(s) ds)^2}}{h \rho(t)}\right] \Phi(x) = 0,\]

(3)

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where $h(t)$ is a positive solution of nonoscillatory equation (1) such that $h'(t) \neq 0$ on some interval $[T_0, \infty)$, $R$ and $G$ are defined by
\[
R(t) := r(t)h^2(t)|h'(t)|^{p-2}, \quad G(t) := r(t)h(t)\Phi(h'(t))
\] (4)
and the conditions
\[
\int_{T_0}^{\infty} \frac{dt}{R(t)} = \infty, \quad \liminf_{t \to \infty} |G(t)| > 0
\]
hold. The critical constant of equation (3) is $\mu_0 = \frac{1}{2q}$, where $q$ is the conjugate number to $p$, i.e., $p^{-1} + q^{-1} = 1$. In [3] it is also shown that (3) has for this constant $\mu = \mu_0$ a solution of the asymptotic formula
\[
x(t) = h(t) \left( \int_t^{\infty} R^{-1}(s) \, ds \right)^{\frac{2}{p}} \left( 1 + \mathcal{O}\left( \left( \int_t^{\infty} R^{-1}(s) \, ds \right)^{-1} \right) \right) \quad \text{as} \quad t \to \infty.
\] This result was qualitatively improved in [8], where a more exact formula for the term $1 + \mathcal{O}\left( \left( \int_t^{\infty} R^{-1}(s) \, ds \right)^{-1} \right)$ was given. More precisely, according to [8], equation (3) has a solution of the form
\[
x(t) = h(t) \left( \int_t^{\infty} R^{-1}(s) \, ds \right)^{\frac{2}{p}} \log L(t),
\] (5)
where $L(t)$ is a generalized normalized slowly varying function of the form
\[
L(t) = \exp \left\{ \int_t^{\infty} \frac{\varepsilon(s)}{R(s) \int_s^{\infty} R^{-1}(\tau) \, d\tau} \, ds \right\}
\]
and $\varepsilon(t) \to 0$ for $t \to \infty$.

Denote $\gamma_p = \left( \frac{p-1}{p} \right)^p$. For the special case $r(t) \equiv 1$, $c(t) = \gamma_p t^{-p}$ and $h(t) = t^{\frac{p-1}{p}}$ the conditionally oscillatory equation (3) with $\mu = \frac{1}{2q}$ becomes the Euler-Weber (or alternatively Riemann-Weber) half-linear differential equation
\[
(\Phi(x'))' + \left[ \frac{\gamma_p}{tp} + \frac{\mu_p}{tp \log^2 t} \right] \Phi(x) = 0
\] (6)
with the critical coefficient $\mu_p = \frac{1}{2q} \left( \frac{p-1}{p} \right)^{p-1}$. Asymptotics of solutions of (6) were firstly studied by Elbert and Schneider in [4], where equation (6) was considered as a perturbation of the half-linear Euler equation
\[
(\Phi(x'))' + \frac{\gamma_p}{tp} \Phi(x) = 0.
\]

In [4] asymptotic formulas of two linearly independent solutions of (6) were given and these results were later improved and discussed in terms of varying functions in [9] and [10]. According to these papers, equation (6) has a pair of linearly independent solutions of the forms
\[
x_1(t) = t^{\frac{p-1}{p}} \log t \frac{1}{t} L_1(t), \quad x_2(t) = t^{\frac{p-1}{p}} \log t \frac{1}{t} (\log \log t) \frac{1}{t} L_2(t),
\] (7)
where $L_1(t)$ is a normalized slowly varying function in the form
\[
L_1(t) = \exp \left\{ \int_t^{\infty} \frac{\varepsilon_1(s)}{s \log s} \, ds \right\}.
with \( \epsilon_1(t) \to 0 \) as \( t \to \infty \) and \( L_2(t) \) is a function in the form

\[
L_2(t) = \exp \left\{ \int t \frac{\epsilon_2(s)}{s \log s \log(\log s)} \, ds \right\} \quad \text{with} \quad \epsilon_2(t) = o(\log(\log t)).
\]

Getting motivated by the results for the Euler-Weber equation, the aim of this paper is to give the asymptotic formula of the second solution of equation (3), linearly independent to (5).

### 2. Preliminaries

One of the basic concepts of the half-linear theory is the Riccati technique, which relates a solution of half-linear equation (1) to a solution of a linear first order differential equation. More precisely, if \( x \) is an eventually positive or negative solution of the nonoscillatory equation (1) on some interval of the form \([T_0, \infty)\), then \( w(t) = r(t) \Phi \left( \frac{\xi}{x} \right) \) solves the Riccati type equation

\[
w' + c(t) + (p - 1)r^{1-q}(t)|w|^q = 0.
\]

Conversely, having a solution \( w(t) \) of (8) for \( t \in [T_0, \infty) \), the corresponding solution of (1) can be expressed as

\[
x(t) = C \exp \left\{ \int t r^{1-q}(s) \Phi^{-1}(w) \, ds \right\},
\]

where \( \Phi^{-1} \) is the inverse function of \( \Phi \) and \( C \) a constant. This means that similarly as in the linear oscillation theory, the nonoscillation of equation (1) is equivalent to the solvability of a Riccati type equation (8) (for details see [1]).

Using the concept of perturbations it is convenient to deal with the so called modified (or generalized) Riccati equation (for details see e.g. [2] and [3]). Let \( h \) be a positive solution of (1) and \( w_h(t) = r(t) \Phi \left( \frac{\xi}{h} \right) \) be the corresponding solution of the Riccati equation (8). Let us consider another nonoscillatory equation

\[
\left( r(t) \Phi(x') \right)' + C(t) \Phi(x) = 0
\]

and let \( w(t) \) be a solution of the Riccati equation associated with (9). Let us denote \( v(t) = (w(t) - w_h(t))h^p(t) \). Then the nonoscillation of equation (1) is equivalent also to the solvability the modified Riccati equation

\[
v' + (C(t) - c(t))h^p + (p - 1)r^{1-q}h^{-q}|G|^q F \left( \frac{\xi}{G} \right) = 0,
\]

where \( G(t) \) is defined by (4) and

\[
F(u) = |u + 1|^q - qu - 1. \tag{11}
\]

Now let us recall the terminology of regularly and slowly varying functions in the sense of Karamata (see [5, 6, 7] and the references therein).

Let a continuously differentiable function \( Q(t) : [T_0, \infty) \to (0, \infty) \) be such that

\[
Q'(t) > 0 \quad \text{for} \quad t \geq T_0, \quad \lim_{t \to \infty} Q(t) = \infty
\]

and let \( g(t), \epsilon(t) \) be some measurable functions satisfying

\[
\lim_{t \to \infty} g(t) = g \in (0, \infty) \quad \text{and} \quad \lim_{t \to \infty} \epsilon(t) = \rho \in \mathbb{R}.
\]
A positive measurable function \( f(t) \), such that \( f \circ Q^{-1} \) is defined for all large \( t \), and which can be expressed in the form

\[
f(t) = g(t) \exp \left\{ \int_{t_0}^{t} \frac{Q'(s)e(s)}{Q(s)} \, ds \right\}, \quad t \geq T_0
\]

for some \( T_0 > t_0 \), is called generalized regularly varying function of index \( \nu \) with respect to \( Q \) (the notation \( f \in RV_Q(\nu) \) is then used). If \( g(t) \equiv g \) (a constant), \( f(t) \) is called normalized regularly varying function. For \( \nu = 0 \) the terminology (normalized) slowly varying function is used.

3. Main result

As a main result we introduce the asymptotic formula for the second solution of equation (3) with the critical coefficient \( \mu = \frac{1}{2q} \), which is linearly independent to the one given by (5). The following statement is also the answer to the open problem conjecturing such result presented in [8].

**Theorem 1.** Let \( \mu = \frac{1}{2q} \). Suppose that (1) is nonoscillatory and possesses a positive solution \( h(t) \) such that \( h'(t) \neq 0 \) for large \( t \) and let conditions

\[
\int_{t_0}^{\infty} \frac{dt}{R(t)} = \infty,
\]

and

\[
\liminf_{t \to \infty} |G(t)| > 0.
\]

hold. Then equation (3) has a solution of the asymptotic formula

\[
x(t) = h(t) \left( \int_{t_0}^{t} R^{-1}(s) \, ds \right)^{\frac{1}{2}} \left( \log \left( \int_{t_0}^{t} R^{-1}(s) \, ds \right) \right)^{\frac{1}{2}} L(t),
\]

where \( L(t) \) is a function in the form

\[
L(t) = \exp \left\{ \int_{t_0}^{t} \frac{\varepsilon(s)}{R(s) R_1^{-1}(s) R_1^{-1}(\tau) \log \left( \int_{\tau}^{t} R_1^{-1}(\tau) \, d\tau \right)} \, ds \right\}
\]

with \( \varepsilon(s) = \Theta \left( \log \left( \int_{t_0}^{t} R^{-1}(s) \, ds \right) \right) \).

**Proof.** The proof proceeds in a similar way as the proof of the similar statement in [8]. First we formulate the modified Riccati equation associated with (3), which reads as

\[
v'(t) + \frac{\mu}{R(t) \left( \int_{t_0}^{t} R^{-1}(s) \, ds \right)^2} + (p-1)q^{-1}(t)h^{-q}(t)|G(t)|qF \left( \frac{v(t)}{G(t)} \right) = 0,
\]

where \( G \) and \( R \) are defined by (4) and \( F \) by (11).

Assumptions (12) and (13) imply the convergence of the integral

\[
\int_{t_0}^{\infty} v^{1-q}(t)h^{-q}(t)|G(t)|qF \left( \frac{v(t)}{G(t)} \right) \, dt,
\]

from which follows that \( v(t) \to 0 \) and \( \frac{v(t)}{G(t)} \to 0 \) for \( t \to \infty \) (see [3]).
Let \( C^0[T, \infty) \) be the set of all continuous functions on the interval \([T, \infty)\) (concrete \(T\) will be specified later) which converge to zero for \( t \to \infty\) and let us consider a set of functions

\[
V = \{ \omega \in C^0[T, \infty) : |\omega(t)| < \varepsilon, t \geq T \},
\]

where \( \varepsilon > 0 \) is such that

\[
(q + 1) \varepsilon < \frac{1}{2}.
\]

It is obvious that the relation

\[
\left( \frac{q}{2} + 1 \right) \varepsilon \leq 1
\]

is implied.

We assume that a solution of the modified Riccati equation (14) is in the form

\[
v(z, t) = \frac{1}{q} \log \left( \int^t R^{-1}(s) \, ds \right) + \frac{2}{q} + z(t),
\]

where \( z \in V \). For brevity let us denote \( J(t) = \int^t R^{-1}(s) \, ds \).

Substituting the derivative \( v'(t) \) into the modified Riccati equation (14) and considering \( r^{1-q}(t) h^{-q}(t)|G(t)|^q R(t) = G^2(t) \), we get the equation

\[
z'(t) + \frac{-z(t) - \frac{2}{q} - \frac{2}{q} \log J(t)}{R(t) J(t) \log J(t)}
- \frac{1}{2q} \log^2 J(t) - z(t) \log J(t) + (p - 1) J^2(t) G^2(t) F \left( \frac{v(z, t)}{G(t)} \right) \log^2 J(t)
+ \frac{E(z, t)}{R(t) J(t) \log J(t)} = 0,
\]

which can be rewritten as

\[
z'(t) + \frac{z(t)}{R(t) J(t) \log J(t)} + \frac{E(z, t)}{R(t) J(t) \log J(t)} = 0
\]

for

\[
E(z, t) = -\frac{2}{q} - \frac{2}{q} \log J(t) - \frac{1}{2q} \log^2 J(t) - 2z(t) - z(t) \log J(t)
+ (p - 1) J^2(t) G^2(t) \log^2 J(t) F \left( \frac{v(z, t)}{G(t)} \right).
\]

Now, let us turn our attention to the behavior of the function \( F(u) \), which plays an important role in estimating of certain needed terms (for details see [3, 8]).

Studying the behavior of \( F(u) \) and \( F'(u) \) for \( u \) in a neighbourhood of 0, we have

\[
F(u) = F''(0) u^2 + F'''(\zeta) u^3 = \frac{q(q-1)}{2} u^2 + \frac{q(q-1)(q-2)}{6} \zeta |1 + \zeta|^{q-3} \text{sgn}(1 + \zeta) u^3,
\]

where \( \zeta \) is between 0 and \( u \). For \( |u| \) small enough there exists a positive constant \( M_q \) such that

\[
\left| F(u) - \frac{q(q-1)}{2} u^2 \right| \leq (q - 1) M_q |u|^3.
\]

Similarly,

\[
F'(u) = F''(0) u + F'''(\zeta) u^2 = q(q-1) u + \frac{q(q-1)(q-2)}{2} |1 + \zeta|^{q-3} \text{sgn}(1 + \zeta') u^2,
\]
where \( \zeta' \) is between 0 and \( u \). Again, for \( |u| \) small enough we have
\[
|F'(u) - q(q - 1)u| \leq 3(q - 1)M_q |u|^2.
\] (19)

The estimate for the function \( E(z, t) \) for \( z \in V \) reads as
\[
|E(z, t)| = \left| -\frac{2}{q} q - \frac{2}{q} \log J(t) - \frac{1}{2q} \log^2 J(t) - 2(z(t) - z(t)) \log J(t) + \frac{2}{q} q^2(t) J^2(t) \log^2 J(t)
\right.
\]
\[
+ (p - 1) J^2(t) G^2(t) \log^2 J(t) F \left( \frac{u}{G} \right) - \frac{2}{q} q^2(t) J^2(t) \log^2 J(t)
\left|
\right.
\]
\[
\leq \left| -\frac{2}{q} q - \frac{2}{q} \log J(t) - \frac{1}{2q} \log^2 J(t) - 2(z(t) - z(t)) \log J(t) + \frac{2}{q} q^2(t) J(t) + \frac{2}{q} q + z(t) ^2
\right|
\]
\[
+ (p - 1) J^2(t) G^2(t) \log^2 J(t) \left( F \left( \frac{v(z, t)}{G(t)} \right) - q(q - 1) \frac{v^2(t)}{G(t)} \right)
\]
\[
\leq \frac{q}{2} |z^2(t)| + \frac{q}{2} M_q \left( \frac{q}{2} \log J(t) + \frac{2}{q} q + z(t) \right)^3
\]
\[
\leq \frac{q}{2} |z^2(t)| + \frac{q}{2} M_q K \left( \frac{q}{2} \log J(t) + \frac{2}{q} q + z(t) \right)^3
\]

where (18) was used for \( u = \frac{q}{2} q \) and \( K := \sup_{t \geq T} |G(t)|^{-1} \) is a finite constant for \( T \) sufficiently large because of (13). According to (12) and using the L’Hospital rule, there exists \( T_1 \) such that the last term in the previous inequality converges and is less than \( \varepsilon^2 \) and therefore
\[
|E(z, t)| \leq \frac{q}{2} q^2 + \varepsilon^2 \leq \varepsilon^2 \left( \frac{q}{2} q + 1 \right) \quad \text{for } t \geq T_1.
\] (20)

Furthermore, for \( z_1, z_2 \in V \) we have
\[
|E(z_1, t) - E(z_2, t)| = \left| (-2 - \log J(t))(z_1 - z_2)
\right.
\]
\[
+ (p - 1) J^2(t) G^2(t) \log^2 J(t) \left[ F \left( \frac{v(z_1, t)}{G(t)} \right) - F \left( \frac{v(z_2, t)}{G(t)} \right) \right]
\left|
\right.
\]
\[
= \frac{q}{2} |z(t)| + (p - 1) J(t) G(t) \log J(t) \left[ F' \left( \frac{v(z, t)}{G(t)} \right) - q(q - 1) \frac{v(z, t)}{G(t)} \right]
\]
\[
\leq \frac{q}{2} |z(t)| + (p - 1) J(t) G(t) \log J(t) \left[ F' \left( \frac{v(z, t)}{G(t)} \right) - q(q - 1) \frac{v(z, t)}{G(t)} \right]
\]
\[
\leq \frac{q}{2} |z(t)| + \left| 3M_q \left( \frac{1}{2} \log J(t) + \frac{2}{q} q + z(t) \right)^2 \right|
\]
where (19) was used. Similarly as in the previous estimate, there exists $T_2$ such that the last term in the last row of the inequality is less than $\varepsilon$ and hence
\[
|E(z_1,t) - E(z_2,t)| \leq \varepsilon(q + 1) \cdot \|z_1 - z_2\| \quad \text{for} \quad t \in [T_2, \infty).
\] (21)

Now, let us consider a function
\[
r(t) = \exp \left\{ \int_0^t \frac{1}{R(t)J(t)\log J(t)} \, ds \right\}.
\]
The equation (17) is then equivalent to
\[
(r(t)z(t))' + \frac{r(t)}{R(t)J(t)\log J(t)} E(z, t) = 0.
\] (22)

Let us define an integral operator $F$ on the set of functions $V$ by
\[
(Fz)(t) = -\frac{1}{r(t)} \int_0^t \frac{r(s)}{R(s)J(s)\log J(s)} E(z, s) \, ds.
\]
Since $J(t) = \int_0^t R^{-1}(s) \, ds$, it is easy to see that
\[
\int_0^t \frac{r(s)}{R(s)J(s)\log J(s)} \, ds = r(t) - c
\]
for some suitable positive constant $c$ and the inequality
\[
\frac{1}{r(t)} \int_0^t \frac{r(s)}{R(s)J(s)\log J(s)} \, ds \leq 1
\]
holds.

For $T = \max\{T_1, T_2\}$, by (20) and (15) we have
\[
|(Fz)(t)| \leq \frac{1}{r(t)} \int_0^t \frac{r(s)}{R(s)J(s)\log J(s)} |E(z, s)| \, ds \leq \left( \frac{q}{2} + 1 \right) \varepsilon^2 \leq \varepsilon,
\]
which means that $F$ maps the set $V$ into itself, and by (21) and (16) we see that
\[
|(Fz_1)(t) - (Fz_2)(t)| \leq \frac{1}{r(t)} \int_0^t \frac{r(s)}{R(s)J(s)\log J(s)} |E(z_1, s) - E(z_2, s)| \, ds
\]
\[
\leq \|z_1 - z_2\| \varepsilon(q + 1) < \frac{1}{2} \|z_1 - z_2\|,
\]
which implies that $F$ is a contraction. Using the Banach fixed-point theorem we can find a function $\tilde{z}(t)$, that satisfies $\tilde{z} = F\tilde{z}$. That means that $\tilde{z}(t)$ is a solution of (22) and also of (17) and $\tilde{v}(t) = \frac{\log J(t) + \frac{q}{2} + \tilde{z}(t)}{J(t)\log J(t)}$ is a solution of (14).

Expressing the solution of the standard Riccati equation for (3) corresponding to the solution $\tilde{v}(\tilde{z}, t)$ of the modified Riccati equation, we have
\[
\tilde{w}(t) = h^{-p}(t)\tilde{v}(\tilde{z}, t) + w_h(t) = w_h(t) \left( 1 + \frac{\tilde{v}(\tilde{z}, t)}{h^p(t)w_h(t)} \right)
\]
\[
= w_h(t) \left( 1 + \frac{\frac{1}{2} \log J(t) + \frac{q}{2} + \tilde{z}(t)}{h^p(t)w_h(t)J(t)\log J(t)} \right) = w_h(t) \left( 1 + \frac{\frac{1}{2} \log J(t) + \frac{q}{2} + \tilde{z}(t)}{G(t)J(t)\log J(t)} \right).
\]
Since a solution of (3) is given by the formula \( x(t) = \exp \left\{ \int_t^a r^{1-q}(s) \Phi^{-1}(\tilde{w}) \, ds \right\} \), we need to express
\[
r^{1-q}(t) \Phi^{-1}(\tilde{w}) = \frac{h'(t)}{h(t)} \left( 1 + \frac{\frac{1}{p} \log J(t)}{G(t)(t) \log J(t)} + \tilde{z}(t) \right)^{q-1}
\]

\[
= \frac{h'(t)}{h(t)} \left( 1 + (q-1) \frac{\frac{1}{p} \log J(t) + \frac{2}{q} + \tilde{z}(t)}{G(t)(t) \log J(t)} + o\left( \frac{\frac{1}{p} \log J(t) + \frac{2}{q} + \tilde{z}(t)}{G(t)(t) \log J(t)} \right) \right)
\]

\[
= \frac{h'(t)}{h(t)} + \frac{\frac{1}{p}}{R(t)J(t)} + \frac{\frac{2}{p}}{R(t)J(t) \log J(t)} + \frac{(q-1)\tilde{z}(t)}{R(t)J(t) \log J(t)} + o\left( \frac{\frac{1}{p} \log J(t) + \frac{2}{q} + \tilde{z}(t)}{R(t)J(t) \log J(t)} \right).
\]

The solution of (3) is in the form
\[
x(t) = \exp \left\{ \log h(t) + \log J^\frac{1}{p}(t) + \log(\log J^\frac{2}{q}(t)) \right\}
\]

\[+ \int_t^a (q-1)\tilde{z}(s) + o\left( \frac{\frac{1}{p} \log J(t) + \frac{2}{q} + \tilde{z}(t)}{R(s)J(s) \log J(s)} \right) \, ds.\]

As \( J(t) = \int_t^a R^{-1}(s) \, ds \), \( \tilde{z} \in V \) and hence \( \tilde{z}(t) \to 0 \) for \( t \to \infty \), the statement of the theorem holds for \( \varepsilon(t) = (q-1)\tilde{z}(t) + o\left( \frac{1}{p} \log J(t) + \frac{2}{q} + \tilde{z}(t) \right) \).

\[\square\]

References


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ON NONLINEAR SPECTRA

N. SERGEJEVA

Dedicated to the memory of prof. RNDr. Pavol Marušiak, DrSc.

Abstract. We construct the Fučík spectrum for some second order boundary value problem with nonzero integral condition. The spectrum contains only finite number of branches.

Introduction

The Fučík spectrum is a set of points \((\mu, \lambda)\) such that the problem 
\[-x'' = \mu x^+ - \lambda x^-, \quad x(0) = 0, \quad x(1) = 0\]
has a nontrivial solution ([1]). This spectrum was studied first in the works of S. Fučík as an object related to “slightly” nonlinear problems. The interest to problems of this type grew up also in connection with the theory of suspension bridges [3]. From the mathematical point of view the Fučík equation became a source of numerous investigations generalizing and refining the results by Fučík.

Let us mention some directions in which the study of Fučík spectra continued. The Fučík spectra have been investigated for the second order equation with different two-point boundary conditions (the Neumann, Sturm-Liouville and periodic ones). Equations with a nonzero right sides were considered in [5]. There are fewer works on the problems of higher order. The third order and fourth order equations of Fučík type were studied by Pope [6], P. Krejčí [2], P. Habets and M. Gaudenzi [4] and others. Some authors (Arias, Campos) have studied non-autonomous Fučík type equations. Of the recent works let us mention also [7].

This paper is devoted to Fučík type problems with the integral condition. The respective spectrum is essentially different from the classical one. It is not yet a countable set of hyperbolas but rather a two-sided wave spreading along the bisectrix of the first quadrant in the parameter \((\mu, \lambda)\) plane. The respective solutions still may have arbitrary number of zeroes. This spectrum was studied and described in the author’s work [8] (see also [9]).

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In the present work we consider the boundary conditions
\[ x(0) = 0, \quad |x'(0)| = 1, \quad \int_0^1 x(t)\,dt = c, \]
where \( c \) is a given constant. The normalization condition is introduced in order to avoid continuous spectra. The analysis of these new spectra is the main goal of this paper.

The paper is organized as follows.

Section 1 is devoted to the Fučík type problem with the zero integral condition. Basic definitions and basic notations are given also.

In Section 2 we present the results on the Fučík spectrum for the problem with nonzero integral conditions. The spectrum is obtained under the normalization condition \( |x'(0)| = 1 \), because otherwise problems may have continuous spectra. This spectrum differs essentially from the known Fučík spectra, it contains only finite number of branches, where the number of branches depends on the value of \( c \). To the best of our knowledge Fučík spectra for such problems have not been considered previously. These are the main results of the work.

In Section 3 we consider a modified problem. The normalization condition \( |x'(0)| = 1 \) is replaced by another one in which \( x'(0) \) depends on a value \( c \) of the integral. We describe properties of the spectrum.

Let us remark that usually two-index description of the branches of the classical Fučík spectrum is employed (and this is enough for \( \lambda \) and \( \mu \) positive). The lower index shows how many zeroes has the respective solution in the interval \((0; 1)\), and the upper index refers to a sign of the derivative of a solution at the initial point. In our notation we use three indices in description of some branches. This complication of notation may be justified by the following reasons. It makes sense in the case of integral boundary conditions to consider solutions which are associated with \((\mu, \lambda)\) also from the second and the fourth quadrants of the plane of parameters. Usually only positive (non-negative) \( \lambda \) and \( \mu \) are employed. Thus the additional “-” sign at the lower index in a description of a branch shows that the respective \( \lambda \) or \( \mu \) is negative. For instance, the notation \( F_{0,-}^{+} \) refers to solutions of the equation \( x'' = -\mu x \), which do not have zeros in the interval \((0, 1]\), which are positive, and \( \mu < 0 \) (these solutions are exponents, not sines and cosines!)

1. About some nonlinear problem with zero integral condition

Consider the problem

\[ -x'' = \mu x^+ - \lambda x^-, \quad \mu, \lambda \in \mathbb{R}, \tag{1} \]
\[ x^+ = \max\{x, 0\}, \quad x^- = \max\{-x, 0\}, \]
with the boundary conditions

\[ x(0) = 0, \quad \int_0^1 x(s)\,ds = 0. \tag{2} \]

**Definition 1.1.** The spectrum is a set of points \((\mu, \lambda)\) such that the problem (1), (2) has nontrivial solutions.
The first result describes decomposition of the spectrum into branches $F^+_i$ and $F^-_i$ ($i = 0, 1, 2, \ldots$) according to the number of zeroes of a solution to the problem (1), (2) in the interval $(0, 1)$.

**Proposition 1.2.** The Fučík spectrum consists of the set of curves

$F^+_i = \{(\mu, \lambda) | x'(0) > 0, \text{ the nontrivial solution of the problem (1), (2) } x(t) \text{ has exactly } i \text{ zeroes in } (0, 1)\}$,

$F^-_i = \{(\mu, \lambda) | x'(0) < 0, \text{ the nontrivial solution of the problem (1), (2) } x(t) \text{ has exactly } i \text{ zeroes in } (0, 1)\}$.

**Theorem 1.3.** The Fučík spectrum $\sum = \bigcup_{i=1}^{+\infty} F^+_i$ for the problem (1), (2) consists of the branches given by

$F^+_1 = F^+_1 \bigcup F^+_2 \bigcup \{(\mu, \lambda) | \mu = (2 + \pi)^2, \lambda = 0\}$,

$F^+_1 = \{(\mu, \lambda) | 2 - \frac{1}{\mu} + \frac{\cos(\sqrt{\lambda - \sqrt{\frac{\pi}{\mu}}})}{\sqrt{\lambda - \sqrt{\frac{\pi}{\mu}}}} = 0, \quad \frac{\pi}{\sqrt{\mu}} < 1, \quad \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}} \geq 1\}$,

$F^+_2 = \{(\mu, \lambda) | 2\frac{1}{\mu} - \frac{2}{\lambda} = 0, \quad \mu > 0, \quad \lambda < 0\}$,

$F^+_2 = \{(\mu, \lambda) | 2i + 1 - 2 - \frac{\cos(\sqrt{\lambda - \sqrt{\frac{\pi}{\mu}}})}{\mu} = 0, \quad i \sqrt{\frac{\pi}{\mu}} + i \sqrt{\frac{\pi}{\lambda}} < 1, \quad (i + 1) \sqrt{\frac{\pi}{\mu}} + i \sqrt{\frac{\pi}{\lambda}} \geq 1\}$,

$F^+_2 = \{(\mu, \lambda) | (i, \mu) \in F^+_1\}$, where $i = 1, 2, \ldots$.

**Proof.** The proof of this theorem for the branches from the first quadrant is given in the present work [9]. The proof for other branches is similar. □

First five branches of the spectrum for the problem (1), (2) are depicted in Figure 1.

2. Spectrum for the Fučík type problem with nonzero integral condition

Consider the equation (1) with the boundary conditions

$x(0) = 0, \quad |x'(0)| = 1, \quad \int_0^1 x(s) ds = c, \quad c \in \mathbb{R}$. (3)

The meaning of the notation of the spectrum is the same as earlier. First we give some lemmas.
Lemma 2.1. The branch $F_{0-}^+$ of the spectrum for the problem (1), (3) exists only for $c > \frac{1}{2}$.

Proof. Let $\mu < 0, \lambda \in \mathbb{R}$, $x'(0) = 1$. Consider the solutions of the problem (1), (3) without zeroes in the interval $(0, 1)$. Such $(\mu, \lambda)$ values belong to the $F_{0-}^+$. We obtain that the problem (1), (3) reduces to the eigenvalue problem $x'' = -\mu x, x(0) = 0, x'(0) = 1, \mu < 0$. The solution of this problem is $x(t) = \frac{1}{\sqrt{-\mu}} \sinh \sqrt{-\mu} t$.

In view of this we obtain
\[ \int_0^1 x(s) ds = \frac{1}{\mu}(1 - \cosh \sqrt{-\mu}). \tag{4} \]

It follows that the integral value tends to $+\infty$ as $\mu$ tends to $-\infty$ and it tends to $\frac{1}{2}$ as $\mu$ tends to $0$ from the left.

The function (4) is the monotone function for negative $\mu$. That is why the branch $F_{0-}^+$ exists only for $c \in \left(\frac{1}{2}, +\infty\right)$.

Lemma 2.2. The branch $F_{0+}^+$ of the spectrum for the problem (1), (3) exists only for $\frac{\pi}{2} \leq c < \frac{1}{2}$.

Proof. Consider the solutions of the problem (1), (3) without zeroes in the interval $(0, 1)$ in the case of $\mu > 0, \lambda > 0$. Such $(\mu, \lambda)$ values correspond to the branch $F_{0+}^+$. We obtain that the problem reduces to eigenvalue problem in
this case. It follows that $\mu \in (0, \pi^2]$. In view of the solution of the problem is
\[ x(t) = \frac{1}{\sqrt{\mu}} \sin \sqrt{\mu} t, \]
we obtain
\[ \int_0^1 x(s) ds = \frac{1}{\mu} (1 - \cos \sqrt{\mu}). \] \hspace{1cm} (5)

In view of it we obtain that the maximal integral value tends to $\frac{1}{2}$ as $\mu$ tends to 0, but the minimal integral value is $\frac{1}{2\pi}$ for $\mu = \pi^2$.

The function (5) is the monotone function in the considered interval $(0, \pi^2]$. That is why the branch $F^{+}_{0+}$ exists only for $c \in \left[ \frac{2}{\pi^2}, \frac{1}{2} \right)$.

**Lemma 2.3.** For any $-\frac{1}{2} < c < \frac{1}{2}$ there exists an integer $i(c)$ value such that the branches $F^{+}_j$ of the spectrum for $j > i$ do not exist.

**Proof.** Now we consider the solutions of the problem (1), (3) with one zero in the interval $(0, 1)$.

First we consider the solutions for $\mu > 0$, $\lambda > 0$. Such $(\mu, \lambda)$ values correspond to the branch $F^{+}_{1+}$.

The solutions of the problem (1), (3) consist of two parts. The first one (in the interval $t \in [0, \tau]$) is the solution of the eigenvalue problem
\[ x'' = -\mu x, \quad x(0) = x(\tau) = 0, \quad x'(0) = 1, \] \hspace{1cm} (6)
where $\tau$ is a zero of the respective solution of the problem (1), (3). We consider the solutions of the problem (6) which have not zeroes in $(0, \tau)$.

But the second one (in the interval $t \in [\tau, 1]$) is the solution of the eigenvalue problem
\[ x'' = -\lambda x, \quad x(\tau) = 0, \quad x'(0) = -1. \] \hspace{1cm} (7)
The solutions of this problem have not zeroes in $(\tau, 1)$.

The calculations show that $\tau = \frac{\pi \sqrt{\mu}}{2}$, the solutions of the problem (6) are the functions $x(t) = \frac{1}{\sqrt{\mu}} \sin \sqrt{\mu} t$, but the solutions of the problem (7) are the functions $x(t) = -\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda} t - \tau)$.

It is clear that the maximal value of integral will be reached if $\tau$ tends to 1, but the minimal one will be reached if $\tau$ tends to 0. That is the maximal value will be obtained if $\mu \to \pi^2$, $\lambda > 0$, but the minimal one if $\mu \to +\infty$, $\lambda \to 0$.

For this reason we obtain that $c \in \left( -\frac{1}{2}, \frac{2}{\pi^2} \right)$. That is why the branch $F^{+}_{1+}$ exists only for such $c$ values.

Now we consider the solutions of the problem (1), (3) with one zero in the interval $(0, 1)$ for $\mu > 0$, $\lambda < 0$. Such $(\mu, \lambda)$ values correspond to the branch $F^{+}_{1-}$.

Similarly as for the previous branch we obtain that the maximal value tends to $\frac{2}{\pi}$ if $\mu \to \pi^2$, $\lambda < 0$, but the minimal one tends to $-\frac{1}{2}$ if $\mu \to +\infty$, $\lambda \to 0$ from the left. It follows that the branch $F^{+}_{1-}$ exists only for $c \in \left( -\frac{1}{2}, \frac{2}{\pi^2} \right)$.

Now we consider the solutions of the problem (1), (3) with two zeroes in the interval $(0, 1)$. Such $(\mu, \lambda)$ values belong to the branch $F^{+}_{2}$.
The solutions of the problem (1), (3) consist of three parts. The first one (in the interval $t \in [0, \tau_1]$) is the solution of the eigenvalue problem

$$x'' = -\mu x, \quad x(0) = x(\tau_1) = 0, \quad x'(0) = 1, \quad (8)$$

where $\tau_1$ is the first zero of the respective solution of the problem (1), (3) and the solutions of the problem (8) which do not have zeroes in $(0, \tau_1)$.

The second one (in the interval $t \in [\tau_1, \tau_2]$) is the solution of the eigenvalue problem

$$x'' = -\lambda x, \quad x(\tau_1) = x(\tau_2) = 0, \quad x'(0) = -1, \quad (9)$$

where $\tau_2$ is the second zero of the respective solution of the problem (1), (3) and the solutions of the problem (9) do not have zeroes in $(\tau_1, \tau_2)$.

The third one (in the interval $t \in [\tau_2, 1]$) is the solution of the eigenvalue problem

$$x'' = -\mu x, \quad x(\tau_2) = 0, \quad x'(0) = 1. \quad (10)$$

The solutions of the problem (10) do not have zeroes in $(\tau_2, 1)$ also.

The calculations show that $\tau_1 = \frac{\pi}{\sqrt{\mu}}, \tau_2 = \frac{\pi}{\sqrt{\lambda}} + \frac{\pi}{\sqrt{\mu}}$, the solutions of the problems (8), (9) and (10) are the functions $x(t) = \frac{1}{\sqrt{\mu}} \sin \sqrt{\mu} t$, $x(t) = -\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda} t - \tau_1)$, $x(t) = \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu} t - \tau_2)$ correspondingly.

Therefore, similar to above, we obtain that the maximal value of integral will be obtained if $\mu \to \pi^2$, $\lambda \to +\infty$, but the minimal one if $\mu \to +\infty$, $\lambda \to \pi^2$.

In view of this we obtain that $c \in \left(-\frac{\pi}{\sqrt{\mu}}, \frac{\pi}{\sqrt{\mu}}\right)$. That is why the branch $F_2^+$ exists only for such $c$ values.

Analogously for another branches of the spectrum we obtain that the branches $F_{2i}^+$ exist only for $c \in \left(-\frac{2}{(i\pi)^2}, \frac{2}{(i\pi)^2}\right)$ and the branches $F_{2i+1}^+$ exist only for $c \in \left(-\frac{2}{(i+1)(\pi)^2}, \frac{2}{(i+1)(\pi)^2}\right)$, where $i = 1, 2, \ldots$. \hfill $\Box$

The similar result is valid for the negative branches also.

**Lemma 2.4.** The branch $F_{-}^0$ of the spectrum for the problem (1), (3) exists only for $c < -\frac{1}{2}$.

**Lemma 2.5.** The branch $F_{-}^0$ of the spectrum for the problem (1), (3) exists only for $-\frac{1}{2} \leq c < \frac{1}{2}$.

**Lemma 2.6.** For any $-\frac{1}{2} < c < \frac{1}{2}$ there exists the integer value $i(c)$ such that the branches $F_j^-$ of the spectrum for $j > i$ do not exist.

**Remark 2.7.** Analogously as in previous lemmas we obtain that the branch $F_{-}^{0+}$ exists only for $c \in \left(-\frac{1}{2}, -\frac{2}{(i\pi)^2}\right)$, but the branch $F_{-0}^-$ exists only for $c \in \left(-\infty, -\frac{1}{2}\right)$, the branches $F_{1i}^+$ and $F_{1i-1}^-$ exist only for $c \in \left(-\frac{2}{(i\pi)^2}, \frac{1}{2}\right)$.

The branches $F_{2i}^-\bar{+}$ exist only for $c \in \left(-\frac{2}{(i\pi)^2}, \frac{2}{(i\pi)^2}\right)$ and the branches $F_{2i+1}^-\bar{+}$ exist only for $c \in \left(-\frac{2}{(i+1)(\pi)^2}, \frac{2}{(i+1)(\pi)^2}\right)$, where $i = 1, 2, \ldots$. \hfill $\square$
Some solutions of the problem (1), (3) with different number of zeroes in the interval \((0, 1)\) are depicted in Figure 2. This Figure is an illustration for Lemmas 2.1 — 2.3.

**Theorem 2.8.** The Fučík spectrum for the problem (1), (3) for any \(c \neq 0, c \neq \pm \frac{1}{2}\) consists of the finite number of branches.

*Proof.* It is a direct consequence of Lemmas 2.1 — 2.6. \(\square\)

**Remark 2.9.** Theorem results are summarized in the Table 1.

<table>
<thead>
<tr>
<th>(c) values</th>
<th>The spectrum consists of</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty; -\frac{1}{2}))</td>
<td>(F_{0-})</td>
</tr>
<tr>
<td>((-\frac{1}{2}; -\frac{2}{π}))</td>
<td>(F_{0-}^+)</td>
</tr>
<tr>
<td>(-i\frac{2}{((i+1)π)^2}; -(i+1)\frac{2}{((i+1)π)^2})</td>
<td>(F_{1-}^-, F_{1-}^+, \ldots, F_{2i-}^-, F_{2i+1}^+)</td>
</tr>
<tr>
<td>(\frac{2}{π}; \frac{2}{π})</td>
<td>(F_{0+}, F_{1+})</td>
</tr>
<tr>
<td>((\frac{1}{2}; +\infty))</td>
<td>(F_{0-}^+)</td>
</tr>
</tbody>
</table>

*Table 1.* Theorem 2.8 results.
Theorem 2.10. The branches of the Fučík spectrum for the problem (1), (3) are given by such equations

\[ F_{0+}^+ = \left\{ (\mu, \lambda) \left| 1 - \cos \sqrt{\frac{\mu}{\lambda}} \frac{\sqrt{\lambda}}{\sqrt{\mu}} = \frac{x'(0)}{x(0)}, \quad 0 < \mu \leq \pi^2, \quad \lambda \neq 0 \right. \right\}, \]

\[ F_{0-}^+ = \left\{ (\mu, \lambda) \left| 1 - \cosh \sqrt{\frac{\mu}{\lambda}} \frac{\sqrt{\lambda}}{\sqrt{\mu}} = \frac{x'(0)}{x(0)}, \quad \mu < 0, \quad \lambda \neq 0 \right. \right\}, \]

\[ F_1^+ = F_{1+}^+ \bigcup F_{1-}^-, \]

\[ F_{1-}^+ = \left\{ (\mu, \lambda) \left| \frac{2}{\mu} - \frac{1}{\lambda} + \cos \left( \sqrt{\sqrt{\mu} - \sqrt{\frac{\pi}{\sqrt{\lambda}}}} \right) \frac{\sqrt{\lambda}}{\sqrt{\mu}} = \frac{x'(0)}{x(0)}, \quad \mu > 0, \quad \lambda < 0 \right. \right\}, \]

\[ F_{2i}^+ = \left\{ (\mu, \lambda) \left| \frac{2i+1}{\mu} - \frac{2}{\lambda} - \cos \left( \sqrt{\sqrt{\lambda} - \sqrt{\frac{\pi}{\sqrt{\mu}}}} \right) \frac{\sqrt{\mu}}{\sqrt{\lambda}} = \frac{x'(0)}{x(0)}, \quad \pi \frac{i+1}{\sqrt{\lambda}} + \pi \frac{i+1}{\sqrt{\mu}} < 1, \quad \pi \frac{i+1}{\sqrt{\lambda}} + \pi \frac{i+1}{\sqrt{\mu}} \geq 1 \right. \right\}, \]

\[ F_{2i+1}^- = \left\{ (\mu, \lambda) \left| \frac{2i+2}{\mu} - \frac{2i+1}{\lambda} - \cos \left( \sqrt{\sqrt{\lambda} - \sqrt{\frac{\pi}{\sqrt{\mu}}}} \right) \frac{\sqrt{\mu}}{\sqrt{\lambda}} = \frac{x'(0)}{x(0)}, \quad \pi \frac{i+1}{\sqrt{\lambda}} + \pi \frac{i+1}{\sqrt{\mu}} < 1, \quad \pi \frac{i+1}{\sqrt{\lambda}} + \pi \frac{i+1}{\sqrt{\mu}} \geq 1 \right. \right\}, \]

\[ F_i^- = \left\{ (\mu, \lambda) \left| (\lambda, \mu) \in F_{3i-1}^+ \right. \right\}. \]

Proof. The proof of this theorem is similar to the proof given in the work [9] for the \((\mu, \lambda)\) from the first quadrant for \(c = 0\). □

Some first branches of the spectrum to the problem (1), (3) are depicted in Figure 3.

3. Spectrum for the Fučík problem with normalization condition, which depends on integral value

Consider the equation (1) with the boundary conditions

\[ x(0) = 0, \quad |x'(0)| = 30\pi c, \quad \int_0^1 x(s)ds = c, \quad c \in \mathbb{R}. \]

(11)

The meaning of the notation of the spectrum is the same as earlier.

Theorem 3.1. The Fučík spectrum for the problem (1), (11) consists of the branches \(F_1^+, \ldots, F_{38}^+, F_{39}^-\), given by equations from Theorem 2.10, where \(|x'(0)| = 30\pi c\).

Proof. Let us change variables in the problem (1), (11) as follows

\[ X(\tau) = \frac{1}{30\pi} x(t). \]
It follows than $X'(\tau) = \frac{1}{30\pi} x'(t)$, $X''(\tau) = \frac{1}{30\pi} x''(t)$.

In view of this we obtain the problem

$$-X'' = \mu X^+ - \lambda X^-;$$

(12)

with boundary conditions

$$X(0) = 0, \quad |X'(0)| = 1, \quad \int_0^1 X(s)ds = \frac{1}{30\pi}.$$  

(13)

The problem (12), (13) is a problem considered in Section 2, where $c = \frac{1}{30\pi}$. That’s why we can use all results about the spectrum from the previous Section. In view of this we obtain that

$$(i + 1)\frac{2}{[(i+1)\pi]^2} \leq \frac{1}{30\pi} < \frac{2}{(i\pi)^2} \quad \text{or} \quad \frac{60-\pi}{\pi} \leq i < \frac{60}{\pi}.$$  

It follows that $i = 19$. The proof of Theorem is complete. \qed

Some first branches of the spectrum to the problem (1), (11) are depicted in Figure 4.

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Figure 4. The Fučík spectrum for the problem (1), (11) in the case for $|x'(0)| = 30\pi c$.

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AN INTRODUCTION TO PRODUCT INTEGRATION ON TIME SCALES

ANTONÍN SLAVÍK

Abstract. This paper represents a short survey of product integration on time scales. We introduce the notion of product ∆-integral of a matrix function defined on an arbitrary time scale, and thus generalize the classical definition of product integral. We summarize its properties and describe its relation to linear systems of dynamic equations. Finally, we generalize the notion of the matrix exponential function.

Introduction

The notion of product integral goes back to V. Volterra (see e.g. [3, 5, 7]). Given a matrix function $A : [a, b] \to \mathbb{R}^{n \times n}$ (where $\mathbb{R}^{n \times n}$ denotes the set of all real $n \times n$ matrices), a partition $a = t_0 < t_1 < \cdots < t_m = b$, and a collection of tags $\xi_i \in [t_{i-1}, t_i]$, $i \in \{1, \ldots, m\}$, we form the product

$$(I + A(\xi_m)(t_m - t_{m-1}))(I + A(\xi_{m-1})(t_{m-1} - t_{m-2})) \cdots (I + A(\xi_1)(t_1 - t_0))$$

(where $I$ stands for the identity matrix). The product integral of $A$ over $[a, b]$, which is usually denoted by the symbol $\prod_a^b (I + A(t) \, dt)$, is defined as the limit of these products as the norm of the partition approaches zero (provided the limit exists).

The interest in product integration stems mainly from the fact that if the matrix function $A$ is a Riemann integrable, then the indefinite product integral

$$Y(t) = \prod_a^t (I + A(s) \, ds), \quad t \in [a, b],$$

is continuous and satisfies the integral equation

$$Y(t) = I + \int_a^t A(s)Y(s) \, ds.$$

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Thus, if $A$ is continuous, then $Y'(t) = A(t)Y(t)$, $Y(a) = I$, and $Y$ is a fundamental matrix of the linear system of equations $y'(t) = A(t)y(t)$, where $y : [a, b] \to \mathbb{R}^n$.

In [2] and [4], M. Bohner and G. Guseinov have developed the basic theory of Riemann integration on time scales. Inspired by their methods, we introduce the notion of product integral on time scales. We assume that the reader is familiar with the basic concepts of the calculus on time scales (see e.g. [1] or [2]). The aim of this paper is to provide an introduction to the theory of product integration on time scales; the proofs will be given elsewhere (see [6]).

1. Basic definitions

Let $\mathbb{T}$ be a time scale. We use the symbol $[a, b]$ to denote a compact interval in $\mathbb{T}$, i.e. if $a, b \in \mathbb{T}$, then $[a, b] = \{t \in \mathbb{T}; a \leq t \leq b\}$. The open and half-open intervals are defined in a similar way.

A partition of $[a, b]$ is a finite sequence of points

$$\{t_0, t_1, \ldots, t_m\} \subset [a, b], \quad a = t_0 < t_1 < \ldots < t_m = b.$$ 

Given such a partition, we put $\Delta t_i = t_i - t_{i-1}$. A tagged partition consists of a partition and a sequence of tags $\{\xi_1, \ldots, \xi_m\}$ such that $\xi_i \in [t_{i-1}, t_i)$ for every $i \in \{1, \ldots, m\}$. The set of all tagged partitions of $[a, b]$ will be denoted by the symbol $D(a, b)$. We will always assume that the division points are called $\{t_0, t_1, \ldots, t_m\}$ and the tags $\{\xi_1, \ldots, \xi_m\}$.

If $\delta > 0$, then $D_\delta(a, b)$ denotes the set of all tagged partitions of $[a, b]$ such that for every $i \in \{1, \ldots, m\}$, either $\Delta t_i \leq \delta$, or $\Delta t_i > \delta$ and $\sigma(t_{i-1}) = t_i$. Note that in the latter case, the only way to choose a tag in $[t_{i-1}, t_i)$ is to take $\xi_i = t_{i-1}$.

The following concept of Riemann $\Delta$-integral represents a time scale generalization of the classical Riemann integral (see [2, 4]):

**Definition 1.1.** Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. A bounded function $f : [a, b] \to \mathbb{R}$ is called Riemann $\Delta$-integrable if there exists a number $S \in \mathbb{R}$ with the property that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\left| \sum_{i=1}^{m} f(\xi_i) \Delta t_i - S \right| < \varepsilon$ for every $D \in D_\delta(a, b)$. The number $S$ is called the Riemann $\Delta$-integral of $f$ over $[a, b]$ and we write

$$\int_{a}^{b} f(t) \Delta t = S.$$

The Riemann $\Delta$-integral of a matrix function $A : [a, b] \to \mathbb{R}^{n \times n}$, where $A = \{a_{ij}\}_{i,j=1}^{n}$, is defined in a straightforward way as the matrix

$$\int_{a}^{b} A(t) \Delta t := \left\{ \int_{a}^{b} a_{ij}(t) \Delta t \right\}_{i,j=1}^{n}$$

(provided all the integrals exist).
We now proceed to the notion of product $\Delta$-integral of a matrix function, a concept which represents a time-scale generalization of the classical product integral.

Let $T$ be a time scale and $a, b \in T$, $a < b$. Given a matrix function $A : [a, b] \to \mathbb{R}^{n \times n}$ and a tagged partition $D \in \mathcal{D}(a, b)$, we denote

$$P(A, D) = \prod_{i=m}^{1} (I + A(\xi_i)\Delta t_i) = (I + A(\xi_m)\Delta t_m) \cdots (I + A(\xi_1)\Delta t_1).$$

(The order is important because matrix multiplication is not commutative.)

**Definition 1.2.** A bounded matrix function $A : [a, b] \to \mathbb{R}^{n \times n}$ is called product $\Delta$-integrable if there exists a matrix $P \in \mathbb{R}^{n \times n}$ with the property that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|P(A, D) - P\| < \varepsilon$ for every $D \in \mathcal{D}_\delta(a, b)$. The matrix $P$ is called the product $\Delta$-integral of $A$ over $[a, b]$ and we write

$$\prod_{a}^{b} (I + A(t)\Delta t) = P.$$

We make the agreement that if $a = b$, then $\prod_{a}^{b} (I + A(t)\Delta t) = I$ for every function $A$.

**Remark 1.3.** The previous definition assumes that the space $\mathbb{R}^{n \times n}$ is equipped with a certain norm. Since all norms on a finite-dimensional space are equivalent, the definition does not depend on the choice of a particular norm.

The following theorem says that if $T = \mathbb{R}$, then the product $\Delta$-integral coincides with the classical notion of product integral. On the other hand, if $T = \mathbb{Z}$ (or more generally $T = h\mathbb{Z}$), then the product $\Delta$-integral reduces to a product of certain matrices.

**Theorem 1.4.** Let $a, b \in T$, $a < b$.

(i) If $T = \mathbb{R}$, then $A : [a, b] \to \mathbb{R}^{n \times n}$ is product $\Delta$-integrable if and only if it is product integrable in the classical sense, and then

$$\prod_{a}^{b} (I + A(t)\Delta t) = \prod_{a}^{b} (I + A(t) dt).$$

(ii) If $T = \mathbb{Z}$, then every function $A : [a, b] \to \mathbb{R}^{n \times n}$ is product $\Delta$-integrable and

$$\prod_{a}^{b} (I + A(t)\Delta t) = \prod_{t=b-1}^{a} (I + A(t)).$$
(iii) If $T = h \mathbb{Z}$, then every function $A : [a, b] \to \mathbb{R}^{n \times n}$ is product $\Delta$-integrable and

$$\prod_{t = a}^{b} (I + A(t) \Delta t) = \prod_{t = a/h}^{a/h} (I + A(th) h).$$

2. Product integral properties

It is not immediately obvious whether a given matrix function $A : T \to \mathbb{R}^{n \times n}$ is product $\Delta$-integrable. The following theorem provides a useful sufficient condition; its proof is rather technical and long, but is similar to its classical counterpart.

**Theorem 2.1.** Every Riemann $\Delta$-integrable function is product $\Delta$-integrable.

**Example 2.2.** Every constant function $A$ is Riemann $\Delta$-integrable, and therefore also product $\Delta$-integrable. If $T = \mathbb{Z}$, then (see Theorem 1.4)

$$\prod_{t = a}^{b} (I + A(t) \Delta t) = (I + A)^{(b-a)}.$$

Now, let $T = \mathbb{R}$. For every $m \in \mathbb{N}$, let $D_m$ be a tagged partition that divides $[a, b]$ into $m$ subintervals of length $(b - a)/m$ (the tags can be arbitrary). Then

$$\prod_{t = a}^{b} (I + A \Delta t) = \lim_{m \to \infty} P(A, D_m) = \lim_{m \to \infty} (I + \frac{A(b-a)}{m})^m = e^{A(b-a)}.$$

The next statement is an easy consequence of the previous theorem and the definition of product $\Delta$-integral.

**Theorem 2.3.** If $A : [a, b] \to \mathbb{R}^{n \times n}$ is Riemann $\Delta$-integrable and $c \in [a, b]$, then

$$\prod_{t = a}^{b} (I + A(t) \Delta t) = \prod_{t = c}^{b} (I + A(t) \Delta t) \cdot \prod_{t = a}^{c} (I + A(t) \Delta t).$$

We now turn our attention to the properties of the indefinite product integral. Again, the proof of the following theorem is rather long and technical, but follows its classical version.

**Theorem 2.4.** If $A : [a, b] \to \mathbb{R}^{n \times n}$ is Riemann $\Delta$-integrable, then the indefinite product $\Delta$-integral

$$Y(t) = \prod_{s = a}^{t} (I + A(s) \Delta s), \quad t \in [a, b],$$

is $\Delta$-integrable. It is called the indefinite product integral of $A$.  

\begin{align*}
\prod_{t = a}^{b} (I + A(t) \Delta t) &\quad \text{is product } \Delta \text{-integrable for every } [a, b] \\
\prod_{t = c}^{b} (I + A(t) \Delta t) \cdot \prod_{t = a}^{c} (I + A(t) \Delta t) &\quad \text{is product } \Delta \text{-integrable.}
\end{align*}
is a continuous function, which satisfies

\[ Y(t) = I + \int_a^t A(s)Y(s)\Delta s, \quad t \in [a, b]. \]

**Corollary 2.5.** If \( A : [a, b] \to \mathbb{R}^{n \times n} \) is rd-continuous and \( y_0 \in \mathbb{R}^n \), then the vector function

\[ y(t) = \prod_t^a (I + A(s)\Delta s)y_0 \]

is a solution of the dynamic equation \( y^\Delta(t) = A(t)y(t) \) such that \( y(a) = y_0 \).

### 3. Regressive functions

We now demonstrate that the theory of product integration on time scales leads to a simple proof of the existence-uniqueness theorem for the linear initial-value problem \( y^\Delta(t) = A(t)y(t), \quad y(t_0) = y_0 \), where \( A : \mathbb{T} \to \mathbb{R}^{n \times n} \) is a regressive rd-continuous function. (A different proof without product integration is given in [1].)

**Definition 3.1.** A function \( A : \mathbb{T} \to \mathbb{R}^{n \times n} \) is called regressive if the matrix \( I + A(t)\mu(t) \) is regular for every \( t \in \mathbb{T} \).

In the classical case, the product integral of an arbitrary matrix function is a regular matrix. This is no longer true on a general time scale; to guarantee the regularity, we have to assume that our matrix function is regressive:

**Theorem 3.2.** If \( A : [a, b] \to \mathbb{R}^{n \times n} \) is a regressive Riemann \( \Delta \)-integrable function, then \( \prod_a^b (I + A(t)\Delta t) \) is a regular matrix.

**Definition 3.3.** If \( a < b \), we define \( \prod_a^b (I + A(t)\Delta t) = \left( \prod_a^b (I + A(t)\Delta t) \right)^{-1} \) provided the right-hand side exists.

**Theorem 3.4.** If \( A : \mathbb{T} \to \mathbb{R}^{n \times n} \) is a regressive rd-continuous function, \( t_0 \in \mathbb{T} \) and \( y_0 \in \mathbb{R}^n \), then the vector function

\[ y(t) = \prod_{t_0}^t (I + A(s)\Delta s)y_0 \]

represents a unique solution of the dynamic equation \( y^\Delta(t) = A(t)y(t) \) such that \( y(t_0) = y_0 \).

The monograph [1] contains the following definition: If \( A : \mathbb{T} \to \mathbb{R}^{n \times n} \) is a regressive rd-continuous matrix function and \( t_0 \in \mathbb{T} \), then the initial value problem \( Y^\Delta(t) = A(t)Y(t), \quad Y(t_0) = I \) has a unique solution, which is denoted by \( e_A(\cdot, t_0) \) and called the matrix exponential.
Using product integration, the notion of matrix exponential can be extended to a wider class of all regressive Riemann $\Delta$-integrable functions. The following theorem shows that the generalized matrix exponential inherits some important properties of the matrix exponential as given in [1].

**Theorem 3.5.** Let $T$ be a time scale and $A : T \to \mathbb{R}^{n \times n}$ a regressive Riemann $\Delta$-integrable function. Then the function

$$ e_A(t, t_0) = \prod_{t_0}^{t} (I + A(s)\Delta s), \quad t_0, t \in T, $$

has the following properties:

1. $e_0(t, s) \equiv I$ and $e_A(t, t) \equiv I$,
2. $e_A(\sigma(t), s) = (I + A(t)\mu(t))e_A(t, s)$,
3. $e_A(t, s)^{-1} = e_A(s, t)$,
4. $e_A(t, s)e_A(s, r) = e_A(t, r)$.

Let us finally remark that the whole theory of product $\Delta$-integral presented in this paper can be easily modified to obtain the corresponding notion of product $\nabla$-integral, which is related to the linear dynamic system $y^{\nabla}(t) = A(t)y(t)$.

**References**


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